Session 1A9
Novel Methods for Solving the Forward and Inverse Problems of Radiative Transport

Analytical Cumulant Solution of the Radiative Transfer Equation for Light Scattering in Turbid Media

W. Cai, **M. Xu**, **X. Ni**, **and R. R. Alfano**
City University of New York, USA

We will discuss an analytical solution of the time-dependent radiative transfer equation in an infinite uniform medium with an arbitrary phase function using cumulant expansion, and compare the theoretical results with the Monte Carlo simulation and experiments.

The expression of the exact spatial cumulants of light distribution function, up to an arbitrary high order, at different angle and time have been derived. We plan to briefly review this derivation. The first cumulant represents the center of the distribution, and the second cumulant represents half-width of spread of the distribution, which can be fast and exactly calculated using the analytical expression. The photon distribution function is expressed by a Gaussian distribution for late times and for backscattering, or by a reshaped distribution for transmission at early times, with exact center and exact half-widths. The computed time-resolved profiles match with that of the Monte Carlo simulation.

The analytical cumulant approach is extended for solution of the polarized (vector) radiative transfer equation. Our computation shows that in the backscattering case circular polarization helicity flips with increase of the size of scatters. These results are compared with our recent experimental results using circular polarization. Approach for including the semi-infinite and slab boundaries is tested, and the results are shown.

Radiative Transport in Rotated Reference Frames

G. Y. Panasyuk, J. C. Schotland, and V. A. Markel

University of Pennsylvania, USA

We have proposed a novel method for solving the linear radiative transport equation (RTE) in a three-dimensional macroscopically homogeneous medium. The method utilizes the concept of locally rotated reference frames and can be used with an arbitrary phase function of a random medium consisting of spherically-symmetrical microscopical scatterers. The angular dependence of the specific intensity written in the spatial Fourier representation is obtained as an expansion into spherical functions defined in reference frames whose z-axes coincide with the direction of the Fourier vector k. In practice, this expansion is truncated at the maximum order $l_{\max}$. Boundary conditions have been considered in the slab and half-space geometries.

We have applied the method to (i) calculation of the RTE Green's function for different values of the absorption and scattering coefficients, μ_a and μ_s, and the asymmetry parameter g within the Henyey-Greenstein model for the phase function [1, 2], and (ii) to generation of forward data for an inverse problem of optical tomography [3]. In particular, we have demonstrated in [3] that the spatial resolution of images obtained in optical tomography is not limited to the fundamental length scale of one transport mean free path.

Angular dependence of the specific intensity I due to a unidirectional point source is illustrated in the figure (see caption for numerical values of relevant parameters). An accurate value of the specific intensity at $\theta \approx \pi/2$ ($\approx 10^{-3}$ relative error) was obtained at $l_{\max} = 34$. Note that for smaller values of $l_{\max}$, such as $l_{\max} = 10$, the result is still grossly inaccurate and can even be negative.

REFERENCES

1. Markel, V. A., "Modified spherical harmonics method for solving the radiative transport equation," *Waves in Random Media*, Vol. 14, L14, 2004.
2. Panasyuk, G. Y., J. C. Schotland, and V. A. Markel, "Radiative transport equation in rotated reference frames," preprint at http://arxiv.org/abs/math-ph/0505054.
3. Panasyuk, G. Y., J. C. Schotland, and V. A. Markel, "Superresolution and corrections to the diffusion approximation in optical tomography," *App. Phys. Lett.* Vol. 87, 101111, 2005.

Numerical Solution of the Radiative Transport Equation for Modulated Imaging of Tissues

A. D. Kimn
University of California, USA

We develop a novel numerical method to solve the radiative transport equation in a slab with an inhomogeneous absorption coefficient for modulated imaging of tissues. In modulated imaging of tissues, the incident light is modulated spatially permitting the sampling of the solution in the spatial frequency domain. We use the iterated source method to handle the integral operator. For the angle variables, we use the discrete ordinate method. We use a Fourier pseudo-spectral method to treat the transverse spatial variables. We use a semi-implicit method to solve the resulting discretized system of equations. We show numerical results using this method of direct images using spatially modulated illumination.

Inverse Transport with Diffusion-type Measurements

G. Bal
Columbia University, USA

Inverse problems in optical tomography consist of reconstructing the optical properties of tissues from boundary measurements of photon intensities that are usually averaged in the angular variable, e.g., when only currents are available at the domain boundary. Whereas an fairly extensive mathematical theory exists on the reconstruction of optical parameters from full (phase-space) measurements in transport theory, very little is known about reconstructions from angularly averaged measurements. We will review recent results obtained recently on the subject.

Hybrid Solution Methods for the Radiative Transport Equation

S. Arridge and M. Schweiger
University College London, UK

T. Tarvainen, V. Kolehmainen, M. Vauhkonen, and J. Kaipio
University of Kuopio, Finland

Optical Tomography based on the Diffusion Approximation (DA) is a well established field of research. Despite the success of the DA in modelling optically thick regions, it is well known that under certain conditions it is no longer valid, in particular, near sources and interfaces, and in low scattering or high absorption regions. Under these circumstances, attention has turned to the more accurate, and computationally more expensive Transport Equation, which in optics is usually referred to as the Radiative Transfer Equation (RTE). A very large body of literature exists for solving this equation with differing computational complexity.

A general method for the RTE involves the representation of the angular variation as well as the spatial variation in terms of an implicit or explicit basis. Full solutions to the transport equation which require a large number of angular basis functions become prohibitively expensive. However, the detailed representation of the angular variation may not be required over the whole domain. In this paper we consider a system where the angular order of basis is adapted, leading to a reduction in the overall system size.

We compare some alternative strategies:

1. a hybrid DA/RTE approach where two different models are explicitly developed and coupled with an interface condition

2. a variable order basis methods where the angular variation is developed in orthogonal basis functions. The coupling of different orders is achieved by truncating the basis expansion at different orders, which imposes a Dirichlet condition on a set of implicit interfaces.

We show results in particular for the parts of domains close to sources, and to non-scattering void regions.

Fourier-Laplace Structure of the Inverse Medium Problem for the Radiative Transfer Equation

J. C. Schotland
University of Pennsylvania, USA

There has been considerable recent interest in the inverse problem of optical tomography. The usual approach to this problem makes use of the diffusion approximation (DA) to the radiative transfer equation (RTE). Within the accuracy of the DA, it is possible to formulate the linearized inverse problem in terms of the inversion of a suitably defined Fourier-Laplace transform which relates spatial fluctuations in the optical absorption of a random medium to the intensity of light transmitted through the medium. In this talk I will discuss analogous results which hold beyond the DA. In particular, it is shown that by making use of the plane-wave expansion for the Green's function of the RTE that a generalized Fourier-Laplace structure arises in the inverse medium problem for the RTE. This is joint work with Arnold Kim and Vadim Markel.

A PDE-constrained Optimization Algorithm for Frequency Domain Optical Tomography

A. H. Hielscher, K. Ren, X. Gu, A. K. Klose, and G. Abdoulaev
Columbia University, USA

Diffuse Optical Tomography (DOT) is a fast developing new medical imaging modality that use near-infrared light to probe various sections of the human body. The light from laser diodes is deliver through optical fibers to several locations on the surface of the body part under investigation. Measurements of backreflected and transmitted light intensities at other positions on the surface are recorded and analyzed. The technology for making these measurements is nowadays readily available and has mainly been applied to breast and brain imaging. However, a major challenge that still remains is the development of efficient numerical schemes that transform these data into useful cross-sectional images of the interior.

In this work, we present a novel approach to solving the inverse problems encountered in optical tomography. We have implemented a PDE-constrained optimization method that uses a finite-volume method for the discretization of the frequency-domain radiative transport equation (RTE). The finite-volume discretization gives rise to an algebraic nonlinear programming problem that is solved using the iterative augmented Lagrangian method. By simultaneously updating both radiance and optical properties, the method solves the forward and inverse problems in optical tomography all at once. In this way, the computing time is greatly reduced as compared to traditional unconstrained optimization methods, during which one has to repeatedly solve the forward problem many times. We tested and quantified the performance of the algorithm for various combinations of mesh sizes, noise, regularization parameters, initial guesses, optical properties and measurement geometries. Besides the speed of the code, we compared image qualities by defining a correlation coefficient ρ as well as a deviation factor δ.

In the cases that involve image reconstruction from synthetic measurement data we observe 10-30-fold decrease in computing time for the constrained optimization code compared to the unconstrained optimization code. The regularization parameter β has some influence on the computing time, but with reasonable values on the order of $\beta \sim 10^{-7}$ to 10^{-9}, the computational time changes less than 20%. In general, reconstruction of both absorption and scattering together took longer than reconstructions of only the scattering coefficient or only the absorption coefficient. As expected the correlation coefficients ρ and deviation factors δ worsen as the signal-to-noise ratio decreases. Similarly δ decrease substantially as the (homogeneous) initial guess is not chosen close to the optical properties of the actual background medium. Interestingly ρ is only weakly affected by the initial guess. As long as the optical properties are chosen within 50% of the actual background medium ρ changes by only 10-20%. Finally δ and ρ do not change once the mesh is fine enough so that the average size of finite volumes becomes less than the average scattering mean free path $(1/\mu_s)$. Another positive aspect of the augmented Lagrangian method is that it maintains storage requirements that are comparable to requirements encountered in unconstrained optimization methods.

*This work was supported in part by the National Institute of Arthritis and Musculoskeletal and Skin Diseases (grant no 5R01 AR046255-05) and the National Institute of Biomedical Imaging and Bioengineering (grant no R01 EB001900-01).