Optimized Satellite System-like Data Fitting on a Spherical Shell

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Abstract—The measurement of irradiance on a spherical shell is a common in different fields that goes from geological, biological and many others. Accuracy depends of a judicious use of the sampling, and the last is often defined by technical limitations due to the available infrastructure in each field. Such is the case of the so called planetary array, described as a set of satellite like trajectories on the spherical shell. Both, the data acquisition and the ideal data base, in this case spherical harmonics, may be optimized. The measured points are distributed over maximal circles obtained from the equator, and each other are related by the corresponding rotations. Each circle has 2L+1 equidistant points (the first one and the last one coincides), and the field on the sphere is adjusted by the determination of the harmonic coefficients, as an optimization problem, with Nc equation systems of 2L+1 simultaneous equations for (L+1)^2 variables.

1. Introduction

We had previously analyzed the problem of the irradiance measurement under circular and spherical geometries, considering a uniform sensor detection system [1]. Now, we are trying to measure a field F, on the surface of the Earth using a spherical detection system. The detectors are localized on trajectories, which are obtained by rotations over the equator, as it is shown in Fig. 1, but the arrangement of the detectors is not uniform distributed on the trajectories as in our earlier spherical experiments.

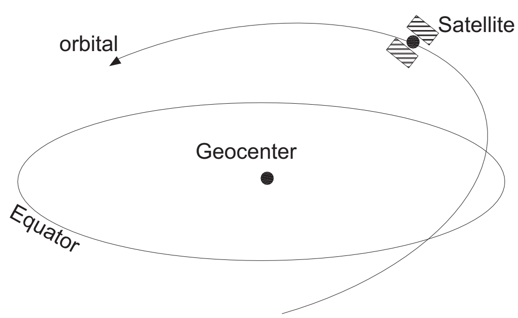

Figure 1: One of the detectors of the proposed spherical detection system.

The analysis of the new system requires the mathematics development that is described as follows:

At first, we consider an observable $F(\theta_n;\phi_n)$ over a determined number of points in the sphere. The points of measurement are distributed over N_c maximal circles, which are obtained since rotations from the equator C^0 (defined by $\theta = \frac{1}{2}\pi$ and $0 \leq \phi < 2\pi$):

$$C^{(\alpha_j,\beta_j,\gamma_j)} := R(\alpha_j,\beta_j,\gamma_j) : C^0 \tag{1}$$

by Euler angles $(\alpha_j,\beta_j,\gamma_j)$, $j = 1, 2, ..., N_c$. The normal lines to these circles are (β_j,γ_j) on the sphere, and its phases are α_j respect to the Greenwich meridian.

Over each circle there are distributed an odd number $2L+1$ of equidistant points, where we measured the value of $F(\theta_n;\phi_n)$. In this way, we can calculate its $2L+1$ Fourier coefficients G_m through the FFT [2]. For the equator case C^0, we calculate:

$$\widetilde{F}^0_m := \frac{1}{\sqrt{2l+1}} \sum_{n=-L}^{L} F(\frac{1}{2}\pi,\phi_n)e^{i(m\phi_n)}, \quad \text{where} \quad \phi_n := \frac{2\pi n}{2L+1}, \tag{2}$$

with $n, m \in \{-L, -L+1, ..., L\}$ module $2L+1$ in the symmetrical interval f cycling. The Fourier synthesis is given by:

$$F(\frac{1}{2}\pi, \phi_n) = \frac{1}{\sqrt{2L+1}} \sum_{m=-L}^{L} \widetilde{F}_m^0 e^{-im\phi_n}. \tag{3}$$

On the maximal circles $C^{(\alpha_j,\beta_j,\gamma_j)}$ we will have the measurements and calculations of the $2L+1$ corresponding coefficients, $\widetilde{F}_m^{(\alpha_j,\beta_j,\gamma_j)}$, $m\,\big|_{-L}^{L}$, $j\,\big|_1^{N_c}$.

The F field over the sphere under measurement has a development in spherical harmonics truncated at L value, given by:

$$F(\theta, \phi) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} F_{\ell,m} Y_{\ell,m} \,. \tag{4}$$

The number of elements in the series is $1+3+5+\cdots+(2L+1) = (L+1)^2$. The question is: How we can calculate the $(L+1)^2$ coefficients of spherical harmonics $F_{\ell,m}$ in terms of the $N_c(2L+1)$ coefficients $\widetilde{F}_m^{(\alpha_j,\beta_j,\gamma_j)}$ obtained over the circles?

In the following section we present the development over the equator, in section 3 we present the rotation over any maximal circle, and in section 4 we compare the obtained results.

2. Development over the Equator

The spherical harmonics are given by:

$$Y_{\ell,m}(\theta, \phi) = (-1)^m \sqrt{(\ell+2)(\ell+m)!(\ell-m)!}\, \frac{e^{im\phi}}{\sqrt{2\pi}} \times \sum_k \frac{(-1)^k (\sin\theta)^{2k+m}}{2^{k+m}(k+m)!k!} \frac{(\cos\theta)^{\ell-2k-m}}{(\ell-2k-m)!} \,. \tag{5}$$

The factorials implies that the addition is over all integers k among $max(-m, 0)$ and $\frac{1}{2}(\ell-|m|)$, the number of elements in the addition is given by $\frac{1}{2}(\ell-|m|)+1$ (where $|x|$ is the integer part of x). Over the equator $\theta = \frac{1}{2}\pi$, the factor $(\cos\theta)^{\ell-2k-m}$ is different of zero, only when its power is zero, *id est* $k = \frac{1}{2}(\ell-m)$ with $\ell-m$ even. Then:

$$Y_{\ell,m}(\frac{1}{2}\pi, \phi) = y_{\ell,m} \frac{e^{im\phi}}{\sqrt{2\pi}}, \quad y_{\ell,m} = y_{\ell,-m}, \tag{6}$$

$$Y_{\ell,m} := \begin{cases} (-1)^{(\ell+m)/2} \frac{\sqrt{(\ell+\frac{1}{2})}(\ell+m)!(\ell-m)!}{2^\ell(\frac{1}{2}[\ell+m])!((\frac{1}{2}[\ell-m])!)} & \ell \pm m \text{ even} \\ \\ 0, & \ell \pm m \text{ odd} \;. \end{cases} \tag{7}$$

Over the equator C^0, the field $F(\theta, \phi)$ development in a different of zero harmonics series (4) is:

$$\begin{aligned} F(\frac{1}{2}\pi, \phi) &= \sum_{\ell=0}^{L} \sum_{m=-\ell}^{L} F_{\ell,m} Y_{\ell,m}(\frac{1}{2}\pi, \phi), && \Big(\text{the harmonics of zero value in } C^0 \text{ are absent} \\ &= \sum_{m=-\ell}^{L} \frac{e^{im\phi}}{\sqrt{2\pi}} \sum_{\substack{\ell=|m| \\ \ell-m \text{ even}}}^{L} F_{\ell,m}\, y_{\ell,m}, && \Big(\text{interchanging additions } \ell \text{ and } m \\ &= \frac{1}{\sqrt{2L+1}} \sum_{m=-\ell}^{L} e^{im\phi} \widetilde{F}_m^0 \,, && \Big(\text{in agreement with (3) considering } \phi_n = 2\pi n/(2L+1). \end{aligned} \tag{8}$$

Comparing the last two terms, related with the coefficients introducing the $2L+1$ measured/calculated data:

$$G_m^0 := \sqrt{\frac{2\pi}{2L+1}}\, \widetilde{F}_m^0 = \sum_{\substack{\ell=|m| \\ \ell-m \text{ even}}}^{L} F_{\ell,m}\, y_{\ell,m}, \quad m\,\big|_{-L}^{L}. \tag{9}$$

Here we have the key relationships among the $2L+1$ coefficients $\{G^0_m\}^L_{m=-L}$ measured/calculated over the equator, and the $(L+1)^2$ coefficients $\left\{F_{\ell,m}\right\}^{\ell\quad L}_{m=-\ell,\ell=0}$ of the spherical harmonic development that we are looking for. The harmonics with $\ell - m$ even are absent, which have one of their nodal circles over the equator.

As an example we consider the case with $L = 3$ Then we have the development with 16 spherical harmonics:

$$Y_{0,0},\ \{Y_{1,m}\}^1_{m=-1}\,,\ \{Y_{1,m}\}^2_{m=-2}\,,\ \text{and}\ \{Y_{1,m}\}^3_{m=-3}$$

with their corresponding coefficients $F_{\ell,m}$.

On the other side, we have the 7 measured/calculated coefficients $\left\{G^0_m\right\}^3_{m=-3}$, $\left\{G^0_m\right\} = \sqrt{2\pi/7}\,\widetilde{F}^0_m$. The equations (9) are then:

$$\begin{array}{llccccccl}
G^0_3 = F_{3,3}\ y_{3,3} & m=3 & & & & \bullet & & \\
G^0_2 = F_{2,2}\ y_{2,2} & m=2 & & & \bullet & \circ & & \\
G^0_1 = F_{1,1}\ y_{1,1} + F_{3,1}\ y_{3,1} & m=1 & & \bullet & \circ & \bullet & & \\
G^0_0 = F_{0,0}\ y_{0,0} + F_{2,0}\ y_{2,0} & m=0 & \bullet & \circ & \bullet & \circ & \longrightarrow \ell & \\
G^0_{-1} = F_{1,-1}\ y_{1,-1} + F_{3,-1}\ y_{3,-1} & m=-1 & & \bullet & \circ & \bullet & & \\
G^0_{-2} = F_{2,-2}\ y_{2,-2} & m=-2 & & & \bullet & \circ & & \\
G^0_{-3} = F_{3,-3}\ y_{3,-3} & m=-3 & & & & \bullet & &
\end{array} \tag{10}$$

The right diagram shows the structure of the present terms $\bullet$ in the truncated series, and the absent ones $\circ$.

The equations used to calculate the 16 [that is, the $(L+1)^2$] coefficients of the harmonic series $F_{\ell,m}$ are divided in three groups: are divided in three groups:

- **Determined**: $F_{3,\pm3}$ and $F_{2,\pm2}$ Always are 4: $F_{L,\pm L}$ and $F_{L-1,\pm(L-1)}$.
- **In linear combination**: $F_{1,1} \leftrightarrow F_{3,1}$, $F_{0,0} \leftrightarrow F_{2,0}$, $F_{1,-1} \leftrightarrow F_{3,-1}$. Generally For $|m| \leq L-2$, there are $F_{\ell,m} \leftrightarrow F_{\ell',m}$ with $0 \leq \ell \leq \ell' \leq L$. In the horizontal line m of the diagram (10), there are a total of $\frac{1}{2}[(L-|m|)]+1$ coefficients in linear combination. There are a total of $\frac{1}{2}(L^2+3L-6)$ $F_{\ell,m}$ coefficients that we known only inside of linear combinations.
- **Undetermined**: $F_{3,\pm2}$, $F_{2,\pm1}$, $F_{1,0}$, and $F_{3,0}$. Generically, they are the known $F_{\ell,m}$ with $\ell - m$ odd, whose spherical harmonics are zero in the equator, and the number of them is $\frac{1}{2}L(L+1)$.

3. Development over the Circle $C^{(\alpha,\beta,\gamma)}$

In this section, we rotate this maximal circle as it is presented in equation (1) in order to obtain the generic circle $C^{(\alpha,\beta,\gamma)}$.

Under the rotation by means of the Euler angles (α, β, γ), the spherical harmonics showed in equation (4) of each ℓ order, is transformed as:

$$Y_{\ell,m}(\theta',\phi') = [R(\alpha,\beta,\gamma) : Y_{\ell,m}](\theta,\phi) = \sum_{m'=-\ell}^{\ell} D^{\ell}_{m,m'}(\alpha,\beta,\gamma) : Y_{\ell,m'}(\theta,\phi), \tag{11}$$

where the rotation of the polar coordinates in the sphere is:

$$\begin{pmatrix} \sin\theta'\cos\phi' \\ \sin\theta'\sin\phi' \\ \cos\theta' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \times \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}. \tag{12}$$

And the coefficients of the linear combination are the functions D of Wigner, $D^{\ell}_{m,m'}(\alpha,\beta,\gamma)$, which is factorized as:

$$D^{\ell}_{m,m'}(\alpha,\beta,\gamma) = e^{-im\alpha} d^{\ell}_{m,m'}(\beta) e^{-im'\gamma}. \tag{13}$$

In terms of phases, for α and γ, and the little-D of Wigner, $d^{\ell}_{m,m'}(\beta)$, given by:

$$d^{\ell}_{m,m'}(\beta)=\sqrt{(\ell+m)!(\ell-m)!(\ell+m')!(\ell-m')!}\sum_k(-1)^{m-m'+k}\frac{(\sin\frac{1}{2}\beta)^{2k+m-m'}}{(k+m-m')!k!}\frac{(\cos\frac{1}{2}\beta)^{2\ell+m'-m-2k}}{(\ell-m-k)!(\ell+m'-k)!}. \quad (14)$$

The k index of the addition takes the integer values among the values: $\max(m'-m,0)\leq k\leq\min(\ell-|m|,\ell-|m'|)$. The little-$d$ coefficients satisfy many relationships, such as:

$$d^{\ell}_{m,m'}(\beta)=d^{\ell}_{m,m'}(-\beta)=d^{\ell}_{-m',-m}(\beta)=(-1)^{m-m'}d^{\ell}_{-m,-m'}(\beta), \quad (15)$$

and they are related with the spherical harmonics by:

$$Y_{\ell,m}(\theta,\phi)=\sqrt{\frac{2L+1}{4\pi}}d^{\ell}_{m,0}(\theta)e^{jm\phi}. \quad (16)$$

They satisfy recurrence S of three terms in m and m':

$$\sqrt{(\ell-m')(\ell+m'+1)}\sin\beta d^{\ell}_{m,m'+1}(\beta)+2(m-m'\cos\beta)d^{\ell}_{m,m'}(\beta)+\sqrt{(\ell+m')(\ell-m'+1)}\sin\beta d^{\ell}_{m,m'-1}(\beta)=0 \quad (17)$$

$$\sqrt{(\ell-m)(\ell+m+1)}\sin\beta d^{\ell}_{m+1,m'}(\beta)-2(m'-m\cos\beta)d^{\ell}_{m,m'}(\beta)+\sqrt{(\ell+m)(\ell-m+1)}\sin\beta d^{\ell}_{m-1,m'}(\beta)=0 \quad (18)$$

The harmonics $Y_{\ell,m}$ are transformed as a column vector. Under the rotation matrix $\|D^{\ell}_{m,m'}\|$, then, the coefficients $F_{\ell,m}$ of the series (4), that we are trying to find, are transformed as a row vector:

$$F'_{\ell,m}=[R(\alpha_j,\beta_j,\gamma_j):F]_{\ell,m}=\sum_{m'=-\ell}^{\ell}F_{\ell,m'}D_{m',m}(\alpha,\beta,\gamma). \quad (19)$$

The equator has been transformed in the circle $C^{(\alpha,\beta,\gamma)}$, nd over it, we make the measurements/calculations of the corresponding coefficients $G^{(\alpha,\beta,\gamma)}_m$, in the same way as in G^0_m, given by equation (10). So, we have, as in (9):

$$G^{(\alpha,\beta,\gamma)}_m=\sum_{\substack{\ell=|m|\\ \ell-m\text{ even}}}^{L}F'_{\ell,m}\,y_{\ell,m}=\sum_{\substack{\ell=|m|\\ \ell-m\text{ even}}}^{L}y_{\ell,m}\sum_{m'=-\ell}^{L}F_{\ell,m'}D^{\ell}_{m',m}(\alpha,\beta,\gamma). \quad (20)$$

The arrangement of the two additions can not be made directly, but in generic form, it can be written in terms of the column vectors:

$$F_{0,0},\ F_{1,\cdot}:=\begin{pmatrix}F_{1,1}\\F_{1,0}\\F_{1,-1}\end{pmatrix},\ F_{2,\cdot}:=\begin{pmatrix}F_{2,2}\\F_{2,1}\\F_{2,0}\\F_{2,-1}\\F_{2,-2}\end{pmatrix},\ \ldots\ F_{\ell,\cdot}:=\begin{pmatrix}F_{\ell,\ell}\\F_{\ell,\ell-1}\\F_{\ell,\ell-2}\\\vdots\\F_{\ell,-\ell}\end{pmatrix}, \quad (21)$$

and the row vectors:

$$D^{\ell}_{\cdot,m}:=(D^{\ell}_{\ell,m}\ D^{\ell}_{\ell-1,m}\ D^{\ell}_{\ell-2,m}\ \ldots\ D^{\ell}_{-\ell,m}). \quad (22)$$

The equations (20), which related to the spherical harmonic coefficients $F_{\ell,m}$ that we are trying to find, and the measured/calculated coefficients $G_m\equiv G^{(\alpha,\beta,\gamma)}_m$ over the maximal circle, are obtained from equation (10) changing $F_{\ell,m}$ by $F'_{\ell,m}$ in accordance with equation (19). The sub-matrix representation can be compared with the case $L=3$:

$$\begin{pmatrix}G^{(\omega)}_3\\G^{(\omega)}_2\\G^{(\omega)}_1\\G^{(\omega)}_0\\G^{(\omega)}_{-1}\\G^{(\omega)}_{-2}\\G^{(\omega)}_{-3}\end{pmatrix}=\begin{pmatrix}0&0&0&y_{3,3}D^3_{\bullet,3}\\0&0&y_{2,2}D^2_{\bullet,2}&0\\0&y_{1,1}D^1_{\bullet,1}&0&0\\y_{0,0}&0&y_{2,0}D^2_{\bullet,0}&0\\0&y_{1,-1}D^1_{\bullet,-1}&0&y_{3,-1}D^3_{\bullet,-1}\\0&0&y_{2,-2}D^2_{\bullet,-2}&0\\0&0&0&y_{3,-3}D^3_{\bullet,-3}\end{pmatrix}\begin{pmatrix}F_{0,\bullet}\\ \\F_{1,\bullet}\\ \\F_{2,\bullet}\\ \\F_{4,\bullet}\end{pmatrix}. \quad (23)$$

The matrix is not square, it has 7 rows and its 4 columns represent the $1 + 3 + 5 + 7 = 16$ columns of the developed matrix, which generically is of $(2L + 1) \times (L + 1)^2$.

When we calculate the elements of the matrix (23), we use the properties $y_{\ell,m} = y_{\ell,-m}$ and the recurrence property mentioned in equation (15).

4. Equation Systems

Over the equator and in respect to the Greenwich meridian, the equation system (23) obtains again it most simple representation, as in equation (10), since $D^{\ell}_{m,m'}(0, 0, 0) = \delta_{m,m'}$ After each rotation, however, the number of determined, in linear combination and the undetermined coefficients are the same (for $L = 3, 4, 6$ and 6). And $2L + 1 < (L + 1)^2$ for $L < 0$.

Considering now, several measurement circles $C^{(\omega_j)}$, $j = 1, 2, \ldots, N_c$, with orientation $\omega_j = (\alpha_j, \beta_j, \gamma_j)$. For each N_c and L, we will have a set of equations of the form (23)–(20):

$$G^{(\omega_j)}_m = \sum_{\substack{\ell=|m| \\ \ell-m \text{ even}}}^{L} y_{\ell,m}\, D^{\ell}_{\bullet,m}(\omega_j)\, F_{\ell,\bullet}\ . \tag{24}$$

They can be written using the double indexes (j and m) and (ℓ and m') for enumerate the arrows and columns of the matrix N_c(2L+1)×(L+1)^2:

$$G = MF \tag{25}$$

where $G = \|G_{j,m}\|\,,\ G_{j,m} = G^{(\omega j)}_m,\ M = \|M_{j,m;\ell,m'}\|\,,\ M_{j,m;\ell,m'} = y_{\ell,m} D^{\ell}_{m',m}(\omega j),\ F = \|F_{\ell,m'}\|\,.$

5. Conclusions

The problem of determining the spherical harmonics of the field $F(\theta,\, \phi)$ over the sphere can be considered as an optimization problem, where there are N_c systems of $2L + 1$ simultaneous equations with $(L + 1)^2$ variables.

If the observations of $F(\theta_n, \phi_n) \Rightarrow F^{(\omega j)}_m \Rightarrow G^{(\omega j)}_m$ are not accurate, we will need to adjust to the harmonic coefficients $F_{\ell,m}$ by minimal square or other algorithm.

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