Approximate Decomposition for the Solution of Boundary Value Problems for Elliptic Systems Arising in Mathematical Models of Layered Structures

Y. Shestopalov and N. Kotik
Karlstad University, Sweden

Abstract—We present an alternative approach to the solution of boundary value problems (BVPs) for elliptic systems arising in mathematical models of layered structures. The main idea of the method is to consider auxiliary problems for differential operators separated componentwise and to reduce them to a sequence of iterative problems such that each can be solved (explicitly) by the Fourier method. The solution sequence is then constructed with the help of a contracting transfer operator evaluated explicitly. This method facilitates both analytic and numerical solutions and can be generalized to more complicated mixed BVPs for semilinear partial differential operators.

1. Introduction

The processes which take place in layered structures may be described in terms of boundary value problems (BVPs) for elliptic systems [1, 2], among them are the Laplace, Helmholtz, and Lamè equations, equipped with appropriate boundary conditions of mixed type, including boundary–value contact problems (BVCPs) formulated and investigated in [3].

The simplest examples of BVPs with boundary conditions of mixed type in electromagnetics and acoustics [1, 2] arise when the Dirichlet (or Neumann) conditions are stated on one part of the boundary and the Neumann (Dirichlet) condition on its complement. Such problem are formulated, e. g., in mathematical models of the wave propagation in transmission lines [1]. A decomposition for the solution to the BVPs for the equation systems can be applied when the differential operator can be separated while the boundary value (trace) operators are mixed componentwise on the boundary. In Section 3 we present an example of such a separation (decomposition).

In this work we present an approach for analytical and numerical solution of BVPs in thin layers based on approximate decomposition. The main idea of this method is to simplify the general BVP and to reduce it to a chain of auxiliary problems and then to a sequence of iterative problems such that each of them can be solved (explicitly) by the Fourier method.

2. Formulation

We present the method for the case of a BVCP [3] for the system of Lamè equations in a thin layer (band) equipped with mixed boundary conditions. To this end, consider an elastic band $S = \{-\infty < x_1 < +\infty,\ 0 < x_2 < h\}$ with Poisson's ratio ν situated on the stiff base $x_2 \equiv 0$. The boundary lines $x_2 = h$ and $x_2 \equiv 0$ are denoted, respectively, by $\mathcal{K}_1$ and $\mathcal{K}_2$ (Fig. 1); $\omega = \bigcup_{m=1}^{N} \omega_m$, where $\omega_m = [\,a_m, b_m]$, is a set of disjoint segments; and $\omega^* = \mathcal{K}_1 \setminus \omega$. Distribution of shearing strains on line $\mathcal{K}_1$, displacements on ω, and elongations on ω^* are given. We denote by u_j and $\mathcal{F}_j$, $(j = 1, 2)$ the displacements and respectively projections of the body forces in directions x_j. The determination of u_j reduces to a mixed BVP [3] for the Lamè equations in S

$$\Delta u_j + k_0 \frac{\partial}{\partial x_j}\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big) = \mathcal{F}_j, \qquad k_0 = \frac{1}{1-2\nu}, \qquad j = 1, 2 \tag{1}$$

with the boundary conditions

$$\begin{aligned} u_2 = 0, \qquad \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} &= 0 && \text{on } \mathcal{K}_2, \\ \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} &= f_1(x_1) && \text{on } \mathcal{K}_1, \\ u_2 &= f_2(x_1) && \text{on } \omega, \\ (k_0 - 1)\frac{\partial u_1}{\partial x_1} + (k_0 + 1)\frac{\partial u_2}{\partial x_2} &= f_3(x_1) && \text{on } \omega^* \end{aligned} \tag{2}$$

and the conditions at infinity

$$\begin{aligned} &\Phi_s(u_1,u_2) = \int\limits_S \Pi_S\, ds < \infty, \\ &\Pi_s = (k_0-1)\Big(\sum_{j=1}^{2}\frac{\partial u_j}{\partial x_j}\Big)^2 + 2\sum_{j=1}^{2}\Big(\frac{\partial u_j}{\partial x_j}\Big)^2 + \Big(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\Big)^2, \end{aligned} \tag{3}$$

BVP (1)–(3) has the unique classical solution if the boundary functions are sufficiently smooth. Namely, the following statement is valid (see [3]):

If the functions $\mathcal{F}_1 \in L_p(S)$, $\mathcal{F}_2 \in L_p(S)$, $f_1 \in L_p(\mathcal{K}_1)$, $f_3 \in L_p(\omega^*)$, $p > 1$ ($f \in L_p(\Omega)$ if $|f|^p$ is integrable over Ω) and function $f_2 \in C^q(\mathcal{K}_1)$, $q \geqslant 3$, is a smooth (q-times continuously differentiable) compactly-supported function with *supp* $f_2 \in \omega$ then problem (1)–(3) is uniquely solvable if and only if

$$\int\limits_{\mathcal{K}_1} f_1\, dx_1 + \int\limits_S \mathcal{F}_1\, dS = 0$$

and the solutions $u_j \in C^2(\Pi_{ah}) \cap C(\bar{\Pi}_{ah})$ in every rectangle $\Pi = \Pi_{ah} = \{(x_1,x_2)\colon 0 < x_1 < a,\ 0 < x_2 < h\}$.

3. Approximate Decomposition

Consider a simplified version of the problem (1)–(3) which will be called *problem A*: body forces $\mathcal{F}_1, \mathcal{F}_2 \equiv 0$; shearing stresses $f_1 \equiv 0$ on $\mathcal{K}_2$; and normal stresses $f_3 \equiv 0$ on ω^*. Consider this problem in a long rectangle Π_{ah} bounded by the curve $\Gamma = \widehat{\mathcal{K}}_1 \bigcup \widehat{\mathcal{K}}_2 \bigcup \mathcal{H}_1 \bigcup \mathcal{H}_2$, where $\widehat{\mathcal{K}}_i = \mathcal{K}_i \bigcap \{0 < x_1 < a\}, (i = 1,2)$; $\widehat{\omega}^* = \omega^* \bigcap \{0 < x_1 < a\}$; $\mathcal{H}_1 = \{\boldsymbol{x} = (x_1,x_2)\colon x_1 = 0, 0 < x_2 < h\}$, $\mathcal{H}_2 = \{\boldsymbol{x} = (x_1,x_2)\colon x_1 = a, 0 < x_2 < h\}$; and $\boldsymbol{u} = (u_1,u_2)$ denotes the vector of displacements. Introduce the trace operators $L^{(1)}$ and $L^{(2)}$ specifying the boundary conditions on $\widehat{\omega}$, $\widehat{\omega}^*$ and Γ:

$$\begin{gathered} L^{(1)}\boldsymbol{u} = \begin{pmatrix} l_{11}^{(1)} & 0 \\ 0 & l_{22}^{(1)} \end{pmatrix}\boldsymbol{u}, \\ l_{11}^{(1)}u_1 = \frac{\partial u_1}{\partial \nu}\ (\boldsymbol{x} \in \Gamma), \qquad l_{22}^{(1)}u_2 = u_2 \qquad (\boldsymbol{x} \in \omega \cup \widehat{\mathcal{K}}_2 \cup \mathcal{H}_1 \cup \mathcal{H}_2) \end{gathered} \tag{4}$$

is the operator of the Neumann–Dirichlet boundary conditions, and

$$\begin{gathered} L^{(2)}\boldsymbol{u} = \begin{pmatrix} l_{11}^{(2)} & l_{12}^{(2)} \\ l_{21}^{(2)} & l_{22}^{(2)} \end{pmatrix}\boldsymbol{u}, \\ l_{11}^{(2)}u_1 = 0, \qquad l_{12}^{(2)}u_2 = \frac{\partial u_2}{\partial \tau} \qquad (\boldsymbol{x} \in \Gamma), \\ l_{21}^{(2)}u_1 = \alpha u_{1,1} \qquad l_{22}^{(2)}u_2 = u_{2,2} \qquad (\boldsymbol{x} \in \widehat{\omega}^*), \end{gathered} \tag{5}$$

where

$$\frac{\partial}{\partial \tau} = \begin{cases} \dfrac{\partial}{\partial x_1}, & \boldsymbol{x} \in \mathcal{K}_1 \bigcup \mathcal{K}_2 \\ \dfrac{\partial}{\partial x_2}, & \boldsymbol{x} \in \mathcal{H}_1 \bigcup \mathcal{H}_2 \end{cases}, \qquad \frac{\partial}{\partial \nu} = \begin{cases} (-1)^i \dfrac{\partial}{\partial x_2}, & \boldsymbol{x} \in \mathcal{K}_i \\ (-1)^i \dfrac{\partial}{\partial x_1}, & \boldsymbol{x} \in \mathcal{H}_i \end{cases}, \qquad \alpha = \frac{k_0+1}{k_0-1}. \tag{6}$$

The operator $L\boldsymbol{u} = L^{(1)}\boldsymbol{u} + L^{(2)}\boldsymbol{u}$ specifies the boundary conditions of problem A in the form $L\boldsymbol{u} = \boldsymbol{f}$, with $\boldsymbol{f} = (0, \widehat{f}_2(\boldsymbol{x}))$ and

$$\widehat{f}_2(\boldsymbol{x}) = \begin{cases} f_2(x_1), & \boldsymbol{x} = (h, x_1) \in \omega, \\ 0, & \boldsymbol{x} \in \Gamma \backslash \omega, \end{cases} \tag{7}$$

being a differentiable function on Γ with a compact support *supp* $f_2 \subseteq \omega$. Introduce matrix differential operators of the system in problem (1)–(3) and problem A and rewrite the latter as

$$\mathcal{D}\boldsymbol{u} = 0, \qquad L\boldsymbol{u} = \boldsymbol{f}, \tag{8}$$

where

$$\mathcal{D} = \Delta + k_0 A, \qquad \Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix},$$
$$\Delta_1 u_1 = (k_0+1)\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2}, \qquad \Delta_2 u_2 = \frac{\partial^2 u_2}{\partial x_1^2} + (k_0+1)\frac{\partial^2 u_2}{\partial x_2^2}, \tag{9}$$
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_2}, \qquad \boldsymbol{f} = (0, \widehat{f}_2(\boldsymbol{x})).$$

Assuming that displacements u_2 are absent on ω^* write problem A in the form

$$\mathcal{D}\boldsymbol{u} = 0, \qquad \widehat{L}\boldsymbol{u} = \boldsymbol{f}, \qquad \widehat{L} = \widehat{L}^{(1)} + \widehat{L}^{(2)}, \tag{10}$$

where $\widehat{L}^{(1)} = \|\widehat{l}_{ii}^{(1)}\|_{i=1,2}$ is defined as in (4) with the only difference that $\widehat{l}_{22}^{(1)} u_2 = \frac{1}{2} u_2$, $\boldsymbol{x} \in \Gamma$, and $\widehat{L}^{(2)}$ has two nontrivial components: $\widehat{l}_{21}^{(2)}$ defined in (5) and $\widehat{l}_{22}^{(2)} u_2 = \frac{1}{2} u_2$, $\boldsymbol{x} \in \Gamma$.

Define the sequence $\{\boldsymbol{u}_n\}$ of vector-functions according to

$$\begin{aligned} &\Delta \boldsymbol{u}_0 = 0, && \widehat{L}^{(1)}\boldsymbol{u}_0 = \boldsymbol{f}_0 = \Big(-\frac{\partial \widehat{f}_2}{\partial x_1}, \widehat{f}_2(x_1)\Big), && x_1 \in \omega, \\ &\Delta \boldsymbol{u}_{n+1} = -k_0 A \boldsymbol{u}_n, && \widehat{L}^{(1)}\boldsymbol{u}_{n+1} = -\widehat{L}^{(2)}\boldsymbol{u}_n, && n = 0, 1, 2, \ldots \end{aligned} \tag{11}$$

The limiting function (if exists) $\boldsymbol{u} = \lim_{n\to\infty} \boldsymbol{u}_n$ (where the limit is determined with respect to an appropriate norm) satisfies (8). In order to prove the existence consider BVP (11) for $\boldsymbol{u}_{n+1} = (u_1^{(n+1)}, u_2^{(n+1)})$. Componentwise, (11) consists of two inhomogeneous BVPs for Poisson equation in the rectangle. The solution to each problem can therefore be obtained as a sum of the corresponding volume and surface (line) potentials. In the vector–operator form the relationship between two intermediate problems (11) can be represented as

$$\boldsymbol{u}_{n+1} = \boldsymbol{K}\boldsymbol{u}_n, \tag{12}$$

where $\boldsymbol{K}$ is a volume–surface integral operator defined in term of the potentials.

Applying the Schauder *a priori* estimates of the solution to BVPs for elliptic PDEs [4, 5], using the explicit form of $\boldsymbol{u}_{n+1}$ and properties of logarithmic and Green's potentials [6, 7], one can show that

$$\|\boldsymbol{u}_{n+1}\|_{C^2(\Pi)} \leqslant M_n\big(\|\boldsymbol{u}_n\|_{C^2(\Pi)} + \|f_2\|_{C^2(\omega)}\big), \qquad n = 1, 2, \ldots, \tag{13}$$

where constant M_n depends on the diameter of Π_{ah} and $M_n \to 0$ if diam $\Pi_{ah} \to 0$. Thus, operator $\boldsymbol{K}$ (12) is a contraction in the space $C^2(\Pi) \cap C(\bar{\Pi})$ of two-component vector-functions if the diameter of set ω, parameter h, and the norm of boundary function f_2 are sufficiently small. This implies the existence of the unique solution $\boldsymbol{u} \in C^2(\Pi) \cap C(\bar{\Pi})$ to problem A.

This approximate decomposition can be applied to the solution of BVPs of the type (1), (2) for semilinear systems with the differential operators $\mathcal{D}\boldsymbol{u} = \triangle \boldsymbol{u} + \mathcal{F}(\boldsymbol{u}, \boldsymbol{u}_{x_1}, \boldsymbol{u}_{x_2}, \boldsymbol{u}_{x_1 x_2})$, where $\mathcal{F}$ is nonlinear with respect to $\boldsymbol{u}$ and $\boldsymbol{u}_{x_i}$. Constructing the iterations similar to (11) or (12) and showing or assuming that the corresponding transfer operator $\boldsymbol{K}$ is contraction, we obtain a recursive procedure (12) to determine displacements $\boldsymbol{u}$.

4. Solution by the Fourier Method

One can obtain explicit solution to every intermediate BVP (11) in the form of Fourier series

$$\begin{aligned} u_2^{(n+1)} &= \sum_{m=1}^{\infty} \sin\frac{\pi m}{a} x_1 \left(d_m \sinh\frac{\pi m\sqrt{k_0+1}}{a} x_2 + e_m \sinh\frac{\pi m}{a\sqrt{k_0+1}} x_2 \right), \\ u_1^{(n+1)} &= \sum_{m=1}^{\infty} \cos\frac{\pi m}{a} x_1 \left(g_m \cosh\frac{\pi m}{a\sqrt{k_0+1}} x_2 + q_m \cosh\frac{\pi m\sqrt{k_0+1}}{a} x_2 \right), \end{aligned} \tag{14}$$

where

$$a_m = -\frac{\sinh\dfrac{\pi m h}{a\sqrt{k_0+1}}}{\sinh\dfrac{\pi m h\sqrt{k_0+1}}{a}} b_m, \qquad b_m = \frac{f_m}{\sinh\dfrac{\pi m h}{a\sqrt{k_0+1}}}, \tag{15}$$

$f_m = \frac{2}{a}\int\limits_0^a f_2(x_1)\sin\frac{\pi m}{a}\,dx_1$ are Fourier coefficients for the function f_2 from boundary condition (2) and

$$\begin{aligned} d_m &= \frac{\sqrt{k_0+1}}{k_0+2}b_m, \qquad & e_m &= \Big(1+\frac{\sqrt{k_0+1}}{k_0+2}\Big)b_m, \\ g_m &= \frac{\sqrt{k_0+1}}{k_0+2}b_m, \qquad & q_m &= \frac{k_0+3}{(k_0+2)\sqrt{k_0+1}}\,a_m b_m \end{aligned} \tag{16}$$

are the Fourier coefficients obtained for (11) on the previous stage n.

Series (14) converge absolutely and uniformly in every rectangle $\Pi^{\delta}_{ah} = \{0 \leqslant x_1 \leqslant a,\, \delta \leqslant x_2 \leqslant h\}$ with $0 < \delta < h$ and admit term-wise differentiation arbitrary number of times. The rate of convergence is exponential.

In view of the explicit solution (14) it is reasonable to specify a boundary function $f_2(x_1)$ in problem A and (7) as a smooth compactly-supported function $f_2 \in C^p(R)$, $p \geqslant 3$, with *supp* $f_2 \in \omega$. One can consider, for example, the case when $f_2(x_1)$ is the so-called hat function of order p (a product of a polynomial in even powers of argument that vanishes at the endpoints of ω and a Gaussian exponent) for which the Fourier coefficients can be calculated explicitly. Such hat functions possess the properties of B–splines; therefore, one can approximate or interpolate a smooth function on the line R with a finite support ω by a finite linear combination of hat functions and apply the approximate decomposition with rapidly converging series solutions to BVPs with virtually arbitrary boundary functions.

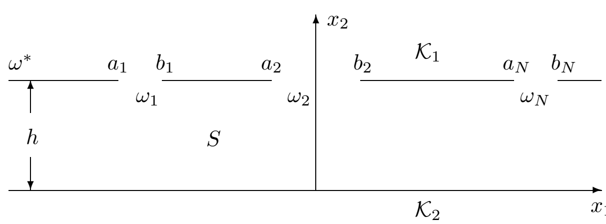

Figure 1: Statement of the problem.

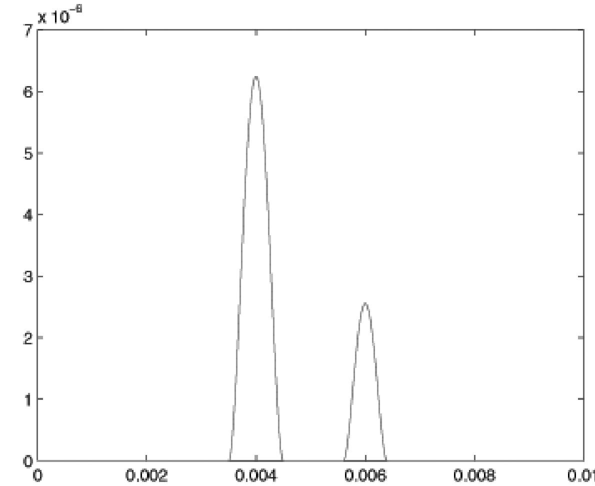

Figure 2: The function f_2.

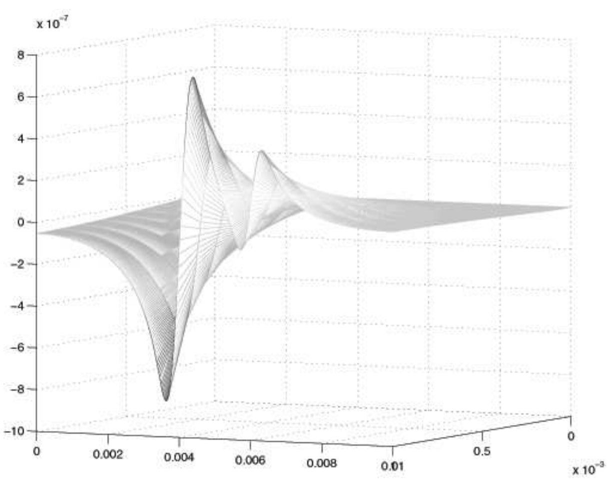

Figure 3: The displacement u_1.

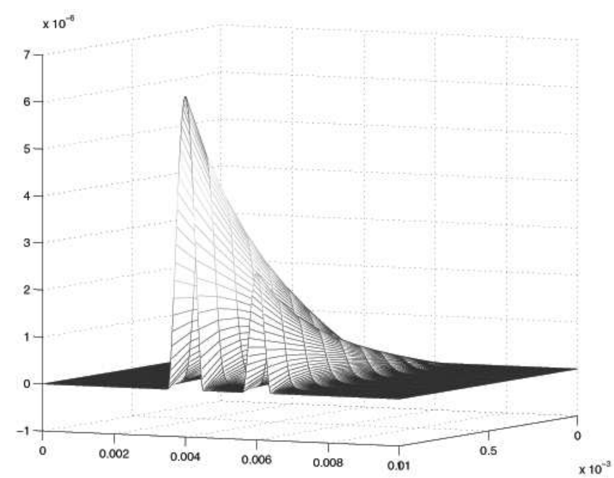

Figure 4: The displacement u_2.

5. Numerical

Let us present some qualitative results of numerical–analytical solution to problem A (a simplified version of (1)–(3) considered in a long rectangle) obtained using approximate decomposition (first iteration); the profiles of boundary displacements are taken as hat functions presented in Fig. 2. Fig.s 3 and 4 show u_1 and u_2 calculated in the case of $a/h = 10$ and two disjoint segments $\omega = \cup_{i=1}^{2}[x_{S_i} - p_i, x_{S_i} + p_i]$.

Values of displacement u_1 in Fig. 3 are zero at $x_{S_{1,2}}$ because these points shift only in x_2-direction; values in the support intervals $(x_{S_1} - p_1, x_{S_1})$ and $(x_{S_2} - p_2, x_{S_2})$ are negative because these points shift in the opposite direction; values in the intervals $(x_{S_1}, x_{S_1} + p_1)$ and $(x_{S_2}, x_{S_2} + p_2)$ are positive because these points also shift in the x_2-direction and take maximum and minimum at the respective points. Function u_2 in Fig. 4 takes only positive values in the intervals $(x_{S_1} - p_1,\ x_{S_1} + p_1)$ and $(x_{S_2} - p_2,\ x_{S_2} + p_2)$, maximum and minimum are at the points x_{S_1} and x_{S_2} respectively.

6. Conclusion

We have developed a method of approximate analytical–numerical solution to BVPs for elliptic system in parallel-plane layers based on decomposition of boundary value conditions. An advantage of the method is the possibility of explicit determination and fast computation and visualization of all components at every point of the layer. The method can be extended to wide families of BVPs using spline-type approximations based on hat functions.

Acknowledgment

The work is supported by the TryckTeknisk Forskning (T2F) program. We would like to thank Dr Magnus Lestelius and Dr Peter Rättö from the Department for Chemistry of the Karlstad University (Sweden) for valuable discussions.

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