High-accuracy Approximation to the Integrated Length of Toroidal/Helical Orbits

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Abstract—The new innovative concept of High Power Microwave (HPM) Amplifier recently introduced, combines Multi-Beam Klystron (MBK) and Electron Storage-Ring (ESR) technologies, by using closed, multi-turn Toroidal/Helical electron-orbits. The selected toroidal/helical orbit-configuration was defined as a parametric space-curve in three dimensional space R^3. We get the parametric equations of that orbit. A high-accuracy approximation has now been obtained for the total length of the Toroidal/Helical orbit, that attains much faster numerical computation. Higher accuracy could be attained by using a higher-order expansion of the orbit-length rate-of-increase.

1. Introduction

High power microwave (HPM) sources are almost always designed as vacuum-electronic devices, and are characterized by the capability of generating output powers in the range of Megawatt to Gigawatt, by using beam-voltages of hundreds of kilovolts, and beam-currents of tens of ampere. HPM sources operate in either of three broadly-defined modes: a) Short Pulse, at pulse lengths of 0.1–10 μs, b) Long Pulse, at pulse lengths of 0.1–10 ms, and c) Continuous Wave (CW). The design of high power microwave (HPM) sources has been gradually evolving during at least the past thirty years, primarily stimulated by applications to high energy charged-particle accelerators, and to directed-energy weapons. A number of classic review-papers document that evolution (see [1, 2]). Currently, high energy charged-particle accelerators use almost exclusively high-power klystron amplifiers [3], that attain peak-powers of hundreds of Megawatt in short-pulse operation, tens of Megawatt in long-pulse operation, and about a Megawatt in CW mode. Such amplifiers provide, while converting DC to microwaves, power-efficiency of 50%–60%, and power-gain of 40 dB–60 dB. The electron beam of klystron amplifiers is sharply bunched by *velocity modulation*, followed by a *drift-space* where the accelerated faster electron catch-up with the decelerated slower electron. The so attained sharp bunching generates, upon the continuous-current electron-beam, the required microwave-frequency component, that is the essential source of the generated high-power microwave output.

Quite recently, a number of multi-beam klystrons (MBK) have been developed experimentally, and at least three different MBK models are already commercially available (see [4–8]). Multi-beam klystrons operate at reduced electron-gun voltage, and higher total beam-current than the single-beam designs, thus preventing occasional destructive gun-diode discharges, and increasing the power-efficiency up to $\sim$ 75%. The power-efficiency (measured as the ratio of output microwave power to input DC power) is however still limited, even in MBK, as it is obviously impossible to extract *all* the microwave energy from a sharply-bunched electron-beam, without having the high space-charge-density of the slowing beam force it into uncontrollable defocusing. All high-power klystron amplifiers include therefore a device known as *the beam dump*, which is a high-volume expansion of the klystron vacuum-enclosure, located beyond the microwave-power extraction-structure, where the not-quite completely energy-depleted electron-beam is collected, while converting (*wasting !*) its residual energy to heat *and* X-rays.

2. Toroidal/Helical Orbits

A new, innovative design of High Power Microwave (HPM) Electron-Beam Amplifier was presented by the Author at the PIERS 2004 Symposium, in Pisa, Italy [14]. That new design has the capability of attaining multi-megawatt output power levels, even in long-pulse, high duty cycle or even continuous wave (CW) operation, with very high efficiency, very high spectral-purity, and very low levels of phase and amplitude noise. The new design was initially conceived as a combination of a multi-beam klystron (MBK), with an Electron Storage Ring (ESR). Very high power-efficiency may be attained by having a sharply-bunched High-Current, Relativistic Electron Beam (HIREB) circulate around a *closed*, *re-entrant* multi-turn orbit, within a strong-focusing, alternating-gradient (AG) magnetic field, generated by an azimuth-periodic lattice of beam-guiding magnetic-dipoles, and magnetic quadrupole lenses. High beam currents may then be attained because, by using a multi-turn *helical* electron-beam orbit, running on the outer surface of a virtual torus-surface, the beam current and the space-charge density in each of the individual orbit-turns can be much lower than in a single-turn orbit. That orbit configuration was initially conceived as a way of increasing the power-efficiency of high-power klystron amplifiers, by eliminating *the beam dump*, and by introducing a mechanism of beam-energy recovery, similar to that of Energy-Recovery Linacs (ERL).

It was soon seen however that the use of such closed multi-turn electron-orbit would essentially reduce the foot-print of the newly-conceived device by a factor in the order of the square of the integer number $\boldsymbol{n}$ of turns, while keeping the total orbit length unchanged, relative to that of a single-turn Electron Storage Ring (ESR).

It was also seen that, by keeping the electron-beam energy always in a relativistic range (such as for instance from 50 Mev to 100 Mev), much higher single-turn beam-current, and much higher total stored beam-energy (expressed in Joule) could be attained under a strong-focusing, alternating-gradient (AGS) magnetic field, while at the same time any partial extraction of microwave-energy from the bunched circulating beam would not appreciably change the electron orbital-frequency. Indeed, the total stored beam-energy (expressed in Joule) is obviously stored in the relativistic $(\boldsymbol{\gamma}-1)\,\boldsymbol{m}_0$ mass-increase of the electrons, multiplied by the square of the constant speed of light. Then, by keeping the electron-energy (expressed in Mev) in a relativistic range, very large amounts of microwave energy (expressed in Joule) could be extracted from the circulating sharply-bunched beam, while hardly changing the electron relativistic velocity-factor $\boldsymbol{\beta}$ ($\boldsymbol{\beta}=\sqrt{(\boldsymbol{\gamma}^2-1)/\boldsymbol{\gamma}^2}$, while $\Delta\,\boldsymbol{E}=\Delta\,\gamma\boldsymbol{m}_0\boldsymbol{c}^2$). As a consequence, such partial microwave-power extraction would hardly change the electron orbit-frequency, provided the beam-energy (expressed in Mev) is kept within a relativistic range, where $\boldsymbol{\beta}$ is a very slow function of $\boldsymbol{\gamma}$. In the light of these considerations, the new HPM amplifier design, that was initially conceived as a combination of the multi-beam klystron (MBK) with an Electron-Storage Ring (ESR), actually appears to perform the function of an Energy-Storage Ring (while still being nevertheless an "ESR"). Quite obviously, in any closed-orbit electron-device, the total orbit length is a parameter of fundamental significance, as it determines both the total electric-charge, and the total beam-energy (expressed in Joule) stored in the orbit, and also determines the orbit-frequency of the electron-bunches. The closed-form exact expression of the orbit-length that was reported in [14], had been obtained by symbolically integrating the rate-of-increase of the orbit-length (also known as "*the speed"!*) as function of the *wrapping-angle* $\boldsymbol{\theta}$, by using *Mathematica*. That expression was however characterized by an extreme degree of complexity, even after being simplified (using the *Mathematica "FullSimplify"* command) from an original a 20-page-long print-out to a single-page expression. The computation-time required by the symbolic integration was only in the order of a minute, even on a modest 166 MHz PC (Dell XPS P166c), but the full simplification of the 20-page-long print-out required four full days, for a total in the order of 96 hours. Even on a modern Workstation, with dual 2.4 GHz Xeon processors (HP xw8000), that simplification requires at least in the order of *six hours*.

3. Orbit Equations

The selected *toroidal/helical* orbit-configuration was defined as a parametric space-curve in three dimensional space R^3, with its Cartesian coordinates being functions of the azimuth-angle $\boldsymbol{\varphi}$ (measured around the torus-axis), and of the *wrapping-angle* $\boldsymbol{\theta}$ (measured around the torus circular cross-section), with the implied condition that the ratio of the two angle periods be rational, such that the orbit closes on itself after an integer number of turns $\boldsymbol{n}$ (for $0\le\boldsymbol{\varphi}\le 2\mathbf{n}\boldsymbol{\pi}$). The parametric equations of that orbit are expressed by:

$$\hat{r}\,(\varphi,\theta)=x\,(\varphi,\theta)\cdot\hat{i}+y\,(\varphi,\theta)\cdot\hat{j}+z\,(\varphi,\theta)\cdot\hat{k} \tag{1}$$

where $\boldsymbol{\varphi}$ is the azimuth angle around the torus-axis, and $\boldsymbol{\theta}$ is the helical *"wrapping angle"* around the torus circular cross-section. The three Cartesian components $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ of the position-vector $\boldsymbol{r}$, and the linear relation between the angles $\boldsymbol{\varphi}$, and $\boldsymbol{\theta}$ are given by:

$$x = (\,R + r\,\cos\theta\,)\,\cos\varphi \tag{2}$$

$$y = (\,R + r\,\cos\theta\,)\,\sin\varphi \tag{3}$$

$$z = r\,\sin\theta \tag{4}$$

$$\theta = \frac{n-1}{n}\,\varphi \tag{5}$$

The closed-form, exact expression given in the original paper [14] for the multi-turn electron-orbit length, as function of the azimuth-angle $\boldsymbol{\varphi}$ around the torus-axis, and of the *wrapping-angle* $\boldsymbol{\theta}$, shows a rather daunting degree of complexity, by including all three Elliptic Integrals: a) of the first kind $\mathbf{E}$, b) of the second kind $\mathbf{F}$, and c) of the third kind $\mathbf{\Pi}$.

4. Orbit-Length Approximations

A high-accuracy approximation has now been obtained for the total length of the Toroidal/Helical orbit, that attains much faster numerical computation. That approximation was obtained by expanding the orbit-length rate-of-increase $\boldsymbol{ds/d\varphi}$ *("the speed" !)* [13] in powers of the torus aspect-ratio $\boldsymbol{c = r\ /\ R}$, and by integrating that expansion term-by-term. The $\mathbf{6}^{th}$-order power-expansion of the orbit-length rate-of-increase obtained is expressed by:

$$\begin{aligned} \frac{d\,s}{d\,\varphi} &= R\sqrt{(1 + c\cos\theta)^2 + c^2\left(\frac{n-1}{n}\right)^2} \\ &\cong w_0 + w_1\,c + w_2\,c^2 + w_3\,c^3 + w_4\,c^4 + w_5\,c^5 + w_6\,c^6 \end{aligned} \tag{6}$$

where the seven $\boldsymbol{w}_i$ expansion-coefficients are given by:

$$w_0 = R \tag{7}$$

$$w_1 = R\cos\theta \tag{8}$$

$$w_2 = \frac{1}{2}\,R\left(\frac{n-1}{n}\right)^2 \tag{9}$$

$$w_3 = -\frac{1}{2}\,R\left(\frac{n-1}{n}\right)^2\cos\theta \tag{10}$$

$$w_4 = \frac{R}{8\,n^2}\left(\frac{n-1}{n}\right)^2\left[4\,n^2\cos^2\theta - (n-1)^2\right] \tag{11}$$

$$w_5 = -\frac{R}{8\,n^2}\left(\frac{n-1}{n}\right)^2\cos\theta\,\left[3\,(2\,n-1) - n^2\left(3 - 4\cos^2\theta\right)\right] \tag{12}$$

$$w_6 = \frac{R}{16}\left(\frac{n-1}{n}\right)^2\left\{\left[\left(\frac{n-1}{n}\right)^2 - 6\cos^2\theta\right]^2 - 28\cos^4\theta\right\} \tag{13}$$

where the *wrapping-angle* $\boldsymbol{\theta}$ is related to $\boldsymbol{\varphi}$ through the linear, rational relation (5): $\boldsymbol{\theta} = [(n-1)/n]\ \boldsymbol{\varphi}$.

Preliminary numerical computations, using $\boldsymbol{n} = 9$ and $\boldsymbol{c} = 0.2$, have shown the residual error of the $\mathbf{6}^{th}$-order expansion of the orbit-length rate-of-increase given in (6) to have a residual error of $-\mathbf{4}\cdot\mathbf{10}^{-6}$ to $+\,\mathbf{3}\cdot\mathbf{10}^{-6}$ across the $0 \leq \boldsymbol{\theta} \leq 2\boldsymbol{\pi}$ range, that is consistently periodic across the whole $\mathbf{0} \leq \boldsymbol{\theta} \leq (\boldsymbol{n} - \mathbf{1})\ \mathbf{2}\boldsymbol{\pi}$ range (Figure 1).

The orbit-length approximate expression, resulting from a term-by-term integration of the $\mathbf{6}^{th}$-order expansion (6) , includes five terms, and is expressed by:

$$s(\varphi) = h_1\,\theta + h_2\,\sin\theta + h_3\,\sin 2\theta + h_4\,\sin 3\theta + h_5\,\sin 4\theta \tag{14}$$

where the five $\boldsymbol{h}_i$ coefficients are functions of the torus aspect-ratio $\boldsymbol{c = r\ /\ R}$, and of the number of orbit-turns $\boldsymbol{n}$:

$$h_1 = R\left\{1+\frac{1}{2}\left(\frac{n-1}{n}\right)^2 c^2+\frac{1}{4}\left(\frac{n-1}{n}\right)^2\left[1-\frac{1}{2}\left(\frac{n-1}{n}\right)^2\right]c^4-\frac{1}{16}\left(\frac{n-1}{n}\right)^2\left[2\left(\frac{n-1}{n}\right)^4-3\frac{(2n-1)^2}{n^4}\right]c^6\right\} \tag{15}$$

$$h_2 = R\left[1 - \frac{1}{2}\left(\frac{n-1}{n}\right)^2 c^2 - \frac{3}{8n^2}\left(\frac{n-1}{n}\right)^2(2n-1)\,c^4\right]c \tag{16}$$

$$h_3 = R\,\frac{1}{16\,n^2}\left(\frac{n-1}{n}\right)^2\left[\left(2 - c^2\right)n^2 + 3\,c^2\,(2\,n-1)\right]c^4 \tag{17}$$

$$h_4 = -R\frac{1}{24}\left(\frac{n-1}{n}\right)^2 c^5 \tag{18}$$

$$h_5 = R\frac{1}{64}\left(\frac{n-1}{n}\right)^2 c^6 \tag{19}$$

The graphic displays of the integrated rate-of-increase $\mathbf{6}^{th}$-order power expansion shown in Figure 2 have been computed for Toroidal/Helical orbits with $\boldsymbol{n} = 9$ turns, and aspect-ratio $\boldsymbol{c = r\ /\ R}$ from 0.1 to 0.4, in steps of 0.1.

A preliminary numerical comparison of the approximate orbit-length expression given in Equation (14), computed using $\boldsymbol{n} = 9$ and $\boldsymbol{c} = 0.2$, has shown the residual error of the approximation to be in the order of

$\pm$ $\mathbf{3} \cdot \mathbf{10}^{-6}$, with a single oscillation period for $\mathbf{0} \leq \boldsymbol{\theta} \leq 2\boldsymbol{\pi}$ (Figure 3). The residual error appears to be a periodic function of the *wrapping-angle* $\boldsymbol{\theta}$, through the whole interval $\mathbf{0} \leq \boldsymbol{\theta} \leq (\mathbf{n} - \mathbf{1})2\boldsymbol{\pi}$. While the orbit-length exact-expression given in [14] shows, for these parameter-values, a periodic discontinuity jump of -6.3836527 at $\boldsymbol{\theta}$-values that are odd-multiples of $\boldsymbol{\pi}$, the approximation given in Equation (14) is completely continuous, and monotonic across the whole $\mathbf{0} \leq \boldsymbol{\theta} \leq (\ \mathbf{n} - \mathbf{1})2\boldsymbol{\pi}$ range, and its computational speed is quite conveniently substantially higher, thus providing the possibility of determining the electron orbital-period, around either a single-turn or an $\boldsymbol{n}$-turn Toroidal/Helical orbit (Figure 3). Quite obviously, higher accuracy could be attained by using a higher-order expansion of the orbit-length rate-of-increase.

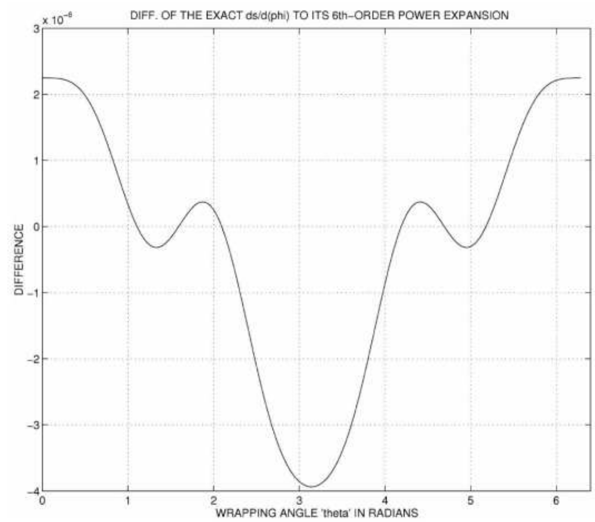

Figure 1: Residual error of the 6th-order expansion of the orbit-length rate-of-increase.

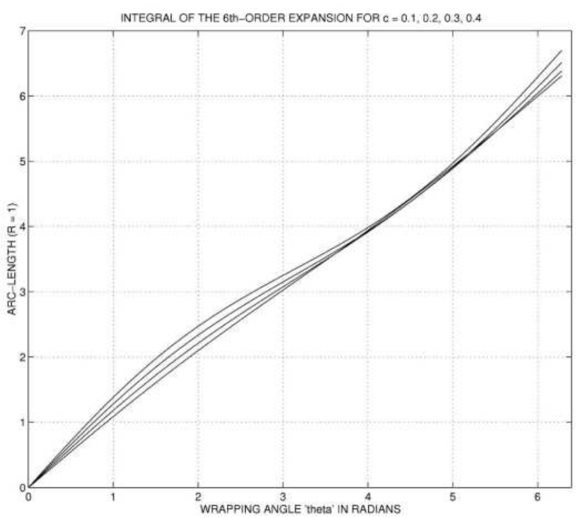

Figure 2: Integrated 6^{th}-order expansion of the toroidal/helical orbit-length rate-of-increase.

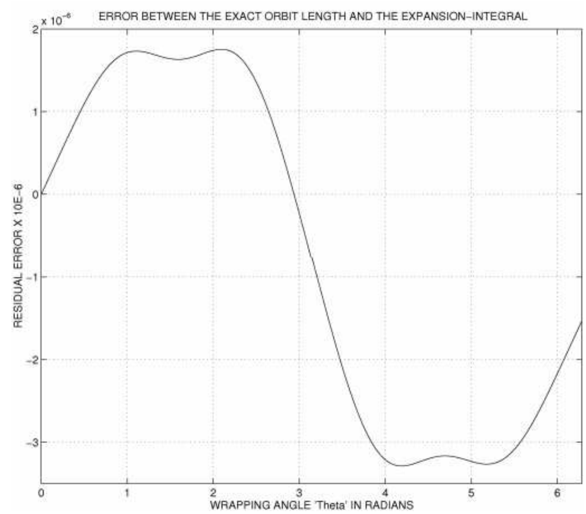

Figure 3: Residual error of the 6^{th}-order power-expansion integral s(φ).

5. Energy Storage

A tentative baseline design of an HPM amplifier as described in [14], has been generated, attempting to match the 1.3 GHz TESLA-Klystron specifications [15]. The virtual torus-surface radii computed are $\boldsymbol{R} = 1444.99$ mm, and respectively $\boldsymbol{r} = 288.998$ mm, while the torus median-circle circumference is $\boldsymbol{L}_c = 9079.15$ mm.

The total 9-turn orbit-length computed $\boldsymbol{L}_t = 83.019$ m shows the use of a $\boldsymbol{n}$-turn Toroidal/Helical orbit to lead to a very compact *"device"* having a surface foot-print $\boldsymbol{n}^2$ times (= 81!) less than a conventional, circular-orbit, electron-storage-ring *"tunnel-installation"*. Further, it appears feasible to have a total of 120 electron-bunch, in 40 sets of three bunch each, nominally spaced by an azimuth increment $\Delta\boldsymbol{\varphi} = 9^o$ around the torus median-circle circumference, so that the bunch-set cyclotron-frequency is only $\boldsymbol{f}_c = 1.3\,\text{GHz}/40 = 0.0325\,\text{GHz} \equiv 32.5\,\text{MHz}$, corresponding to an electron cyclotron period $\boldsymbol{t}_c = 30.7692$ nanosec. The three electron-bunch in each set are then spaced by a nominal wrapping-angle increment $\Delta\theta = 120^o$. Also, the orbit-parameters $\boldsymbol{R}$, $\boldsymbol{r}$, $\boldsymbol{L}_c$, and $\boldsymbol{L}_t$ would hardly change if the electron-energy is always kept sufficiently high, such as from 50 Mev to 150 Mev. Further, it appears also feasible to run a total average beam-current of 9 orbit-turn $\times$ 1.11 kA each = 10 kA total, attaining a circulating electron-beam power of 500–1500 Gw, and a total beam kinetic-energy content between $\boldsymbol{E}_1 = 500\ \boldsymbol{t}_c = 15{,}384.615$ Joule at 50 Mev, and $\boldsymbol{E}_2 = 1500\ \boldsymbol{t}_c = 46{,}153.385$ Joule at 150 Mev. Extracting a partial energy $\Delta\mathbf{E} = \boldsymbol{E}_2 - \boldsymbol{E}_1 = 30{,}769$ Joule, by switching the circulating electron-beam from an accelerating-structure to a microwave-power extraction-structure, would be sufficient to generate a 10 Mw peak-power, 3 msec long microwave pulse, thus exceeding the required TESLA-Collider RF System specification [15] by a factor two in pulse-length. Re-acceleration of the electron-beam could be performed during the pulse-to-pulse 98.5 ms spacing, of the specified maximum 10 Hz pulse-repetition rate. The re-acceleration could be performed at the third sub-harmonic of the required 1.3 GHz output-frequency, by placing the re-acceleration structure along a single orbit-turn, where the azimuth bunch-spacing is $\Delta\varphi = 27^o$.

6. Conclusion

We give an overview on the design of high power microwave (HPM) sources. As a new, innovative design of High Power Microwave (HPM) Electron-Beam Amplifier was presented not long before, a high-accuracy approximation has now been obtained for the total length of the Toroidal/Helical orbit, that attains much faster numerical computation. Higher accuracy could be attained by using a higher-order expansion of the orbit-length rate-of-increase. A tentative baseline design of an HPM amplifier has been generated, attempting to match the 1.3 GHz TESLA-Klystron specifications.

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