Exact Expressions of the Orbit-curvature and Curvature-radius of the Toroidal/Helical Orbits

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Abstract—Closed-from, exact expressions of the Toroidal/Helical orbit-curvature and curvature-radius, have been obtained by first computing the first and second derivatives of the x, y and z components of the orbit position-vector $\mathbf{r}$ as function of the wrapping-angle $\boldsymbol{\theta}$, and by then substituting those derivatives in the general expression of the curvature of a parametric space-curve in three dimensions. Such closed-form exact expressions open the possibility of computing the value of the continuously-evolving dipole magnetic-field $\boldsymbol{B}$, required to guide an electron-beam of known given energy $\boldsymbol{E}$, expressed in Mev , along the given orbit. Additionally, the maximum and minimum values of the dipole magnetic-field were given by performing preliminary numerical computations with $n = 9$ and $c = 0.2$.

1. Introduction

High power microwave (HPM) sources are almost always designed as vacuum-electronic devices, and are characterized by the capability of generating output powers in the range of Megawatt to Gigawatt, by using beam-voltages of hundreds of kilovolts, and beam-currents of tens of ampere. HPM sources operate in either of three broadly-defined modes: a) Short Pulse, at pulse lengths of 0.1–10 μs, b) Long Pulse, at pulse lengths of 0.1–10 ms, and c) Continuous Wave (CW). The design of high power microwave (HPM) sources has been gradually evolving during at least the past thirty years, primarily stimulated by applications to high energy charged-particle accelerators, and to directed-energy weapons. A number of classic review-papers document that evolution (see [1, 2]). Currently, high energy charged-particle accelerators use almost exclusively high-power klystron amplifiers [3], that attain peak-powers of hundreds of Megawatt in short-pulse operation, tens of Megawatt in long-pulse operation, and about a Megawatt in CW mode. Such amplifiers provide, while converting DC to microwaves, power-efficiency of 50%–60%, and power-gain of 40 dB–60 dB. The electron beam of klystron amplifiers is sharply bunched by *velocity modulation*, followed by a *drift-space* where the accelerated faster electron catch-up with the decelerated slower electron. The so attained sharp bunching generates, upon the continuous-current electron-beam, the required microwave-frequency component, that is the essential source of the generated high-power microwave output.

Quite recently, a number of multi-beam klystrons (MBK) have been developed experimentally, and at least three different MBK models are already commercially available (see [4–8]). Multi-beam klystrons operate at reduced electron-gun voltage, and higher total beam-current than the single-beam designs, thus preventing occasional destructive gun-diode discharges, and increasing the power-efficiency up to $\sim$ 75%. The power-efficiency (measured as the ratio of output microwave power to input DC power) is however still limited, even in MBK, as it is obviously impossible to extract *all* the microwave energy from a sharply-bunched electron-beam, without having the high space-charge-density of the slowing beam force it into uncontrollable defocusing. All high-power klystron amplifiers include therefore a device known as *the beam dump*, which is a high-volume expansion of the klystron vacuum-enclosure, located beyond the microwave-power extraction-structure, where the not-quite completely energy-depleted electron-beam is collected, while converting (*wasting !*) its residual energy to heat *and* X-rays.

2. Toroidal/Helical Orbits

A new, innovative design of High Power Microwave (HPM) Electron-Beam Amplifier was presented by the Author at the PIERS 2004 Symposium, in Pisa, Italy [14]. That new design has the capability of attaining multi-megawatt output power levels, even in long-pulse, high duty cycle or even continuous wave (CW) operation, with very high efficiency, very high spectral-purity, and very low levels of phase and amplitude noise. The new design was initially conceived as a combination of a multi-beam klystron (MBK), with an Electron Storage Ring (ESR). Very high power-efficiency may be attained by having a sharply-bunched High-Current, Relativistic Electron Beam (HIREB) circulate around a *closed*, *re-entrant* multi-turn orbit, within a strong-focusing, alternating-gradient (AG) magnetic field, generated by an azimuth-periodic lattice of beam-guiding magnetic-dipoles, and magnetic quadrupole lenses. High beam currents may then be attained because, by using a multi-turn *helical* electron-beam orbit, running on the outer surface of a virtual torus-surface, the beam current and the space-charge density in each of the individual orbit-turns can be much lower than in a single-turn orbit. That orbit configuration was initially conceived as a way of increasing the power-efficiency of high-power klystron amplifiers, by eliminating *the beam dump*, and by introducing a mechanism of beam-energy recovery, similar to that of Energy-Recovery Linacs (ERL).

It was soon seen however that the use of such closed multi-turn electron-orbit would essentially reduce the foot-print of the newly-conceived device by a factor in the order of the square of the integer number $\boldsymbol{n}$ of turns, while keeping the total orbit length unchanged, relative to that of a single-turn Electron Storage Ring (ESR).

It was also seen that, by keeping the electron-beam energy always in a relativistic range (such as for instance from 50 Mev to 100 Mev), much higher single-turn beam-current, and much higher total stored beam-energy (expressed in Joule) could be attained under a strong-focusing, alternating-gradient (AGS) magnetic field, while at the same time any partial extraction of microwave-energy from the bunched circulating beam would not appreciably change the electron orbital-frequency. Indeed, the total stored beam-energy (expressed in Joule) is obviously stored in the relativistic $(\boldsymbol{\gamma}-1)\,\boldsymbol{m}_0$ mass-increase of the electrons, multiplied by the square of the constant speed of light. Then, by keeping the electron-energy (expressed in Mev) in a relativistic range, very large amounts of microwave energy (expressed in Joule) could be extracted from the circulating sharply-bunched beam, while hardly changing the electron relativistic velocity-factor $\boldsymbol{\beta}$ ($\boldsymbol{\beta}=\sqrt{(\gamma^2-1)/\gamma^2}$, while $\Delta\,\boldsymbol{E}=\Delta\,\gamma\boldsymbol{m}_0\boldsymbol{c}^2$). As a consequence, such partial microwave-power extraction would hardly change the electron orbit-frequency, provided the beam-energy (expressed in Mev) is kept within a relativistic range, where $\boldsymbol{\beta}$ is a very slow function of $\boldsymbol{\gamma}$. In the light of these considerations, the new HPM amplifier design, that was initially conceived as a combination of the multi-beam klystron (MBK) with an Electron-Storage Ring (ESR), actually appears to perform the function of an Energy-Storage Ring (while still being nevertheless an "ESR"). Quite obviously, in any closed-orbit electron-device, the local orbit curvature is a parameter of fundamental significance, as it determines the magnetic-field flux-density required in the beam-steering dipole-magnets, and in the beam-focusing quadrupole lenses, as function of the electron-beam energy (expressed in Mev), and also determines the orbit-frequency of the electron-bunches, by determining the orbit curvature-radius. A closed-form, exact expression of the orbit-curvature, has now been obtained by first computing the first and second derivatives of the $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ components of the orbit position-vector $\boldsymbol{r}$ as function of the *wrapping-angle* $\boldsymbol{\theta}$, and by then substituting those derivatives in the general expression of the curvature of a parametric space-curve in three dimensions [13].

3. Orbit Equations

The selected *toroidal/helical* orbit-configuration was defined as a parametric space-curve in three dimensional space (R^3), with its Cartesian coordinates being functions of the azimuth-angle $\boldsymbol{\varphi}$ (measured around the torus-axis), and of the *wrapping-angle* $\boldsymbol{\theta}$ (measured around the torus circular cross-section), with the implied condition that the ratio of the two angle periods be rational, such that the orbit closes on itself after an integer number of turns $\boldsymbol{n}$ (for $0\le\boldsymbol{\varphi}\le 2\mathbf{n}\boldsymbol{\pi}$). The parametric equations of that orbit are expressed by:

$$\hat{r}(\varphi,\theta)=x(\varphi,\theta)\cdot\hat{i}+y(\varphi,\theta)\cdot\hat{j}+z(\varphi,\theta)\cdot\hat{k} \tag{1}$$

where $\boldsymbol{\varphi}$ is the azimuth angle around the torus-axis, and $\boldsymbol{\theta}$ is the helical *"wrapping angle"* around the torus circular cross-section. The three Cartesian components $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ of the position-vector $\boldsymbol{r}$, and the linear relation between the angles $\boldsymbol{\varphi}$, and $\boldsymbol{\theta}$ are given by:

$$x=(R+r\cos\theta)\cos\varphi \tag{2}$$

$$y=(R+r\cos\theta)\sin\varphi \tag{3}$$

$$z=r\sin\theta \tag{4}$$

$$\theta=\frac{n-1}{n}\varphi \tag{5}$$

A 3D display of the selected Toroidal/Helical orbit configuration is shown in Figure 1, including a 3D display of the locus of the moving curvature center (*the "evolute" !*). A 2D display of the corresponding X–Y plane projections is shown in Figure 2.

4. Curvature

Closed-form expressions have now been obtained for the multi-turn electron-orbit *curvature,* and for the orbit *curvature-radius*, as function of the azimuth-angle $\boldsymbol{\varphi}$, and *wrapping-angle* $\boldsymbol{\theta}$. The general expression of the curvature of a space-curve in 3D is [11]:

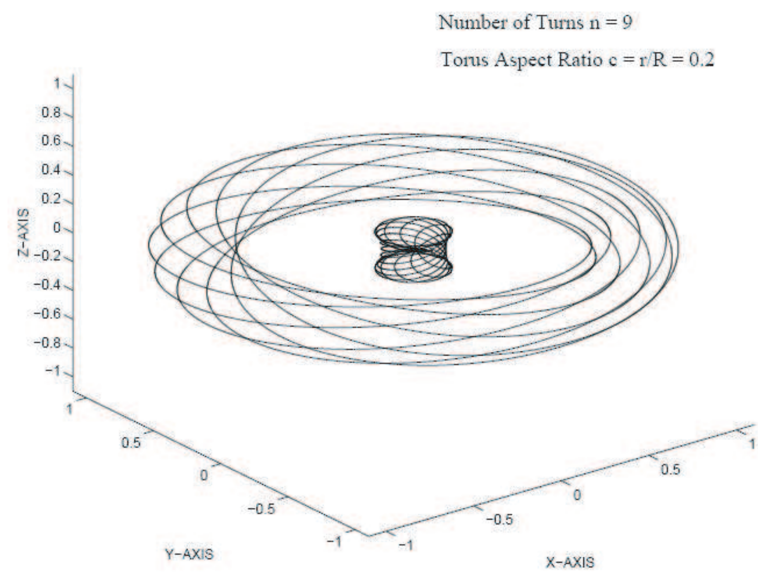

Figure 1: 3D display of the toroidal/helical orbit and of its moving curvature center.

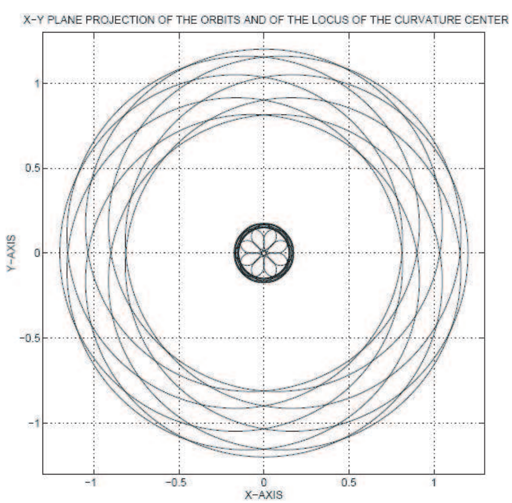

Figure 2: X-Yplane projection of the toroidal/helical orbit and of its moving curvature center.

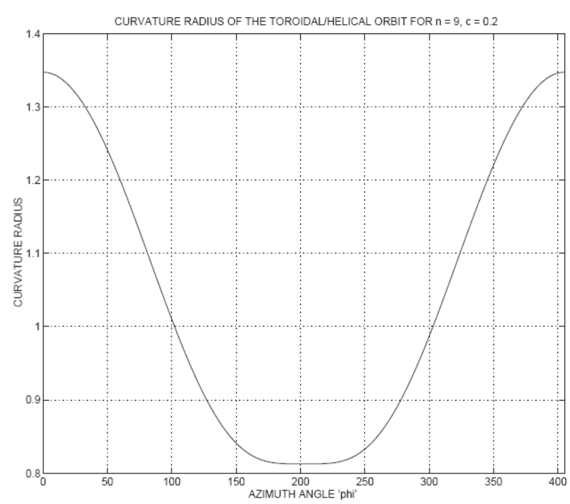

Figure 3: Normalized curvature radius of the toroidal/helical orbit for $n = 9$ and $c = 0.2$.

$$\kappa(\varphi) = \frac{\vec{r'}(\varphi) \times \vec{r''}(\varphi)}{\left\|\vec{r'}(\varphi)\right\|^3} \quad \text{where} \tag{6}$$

$$\vec{r'}(\varphi) = \frac{dx}{d\varphi}\cdot\vec{i} + \frac{dy}{d\varphi}\cdot\vec{j} + \frac{dz}{d\varphi}\cdot\vec{k} \quad \text{and} \tag{7}$$

$$\vec{r''}(\varphi) = \frac{d^2x}{d\varphi^2}\cdot\vec{i} + \frac{d^2y}{d\varphi^2}\cdot\vec{j} + \frac{d^2z}{d\varphi^2}\cdot\vec{k} \tag{8}$$

are the first and second derivatives of the position-vector $\hat{r}(\varphi, \theta) = x(\varphi, \theta) \cdot \hat{i} + y(\varphi, \theta) \cdot \hat{j} + z(\varphi, \theta) \cdot \hat{k}$

The vector-product in the numerator of the general expression (6) is given by:

$$\vec{r}'(\varphi) \times \vec{r}''(\varphi) = \begin{bmatrix} i & j & k \\ \frac{dx}{d\varphi} & \frac{dy}{d\varphi} & \frac{dz}{d\varphi} \\ \frac{d^2x}{d\varphi^2} & \frac{d^2y}{d\varphi^2} & \frac{d^2z}{d\varphi^2} \end{bmatrix} \tag{9}$$

and it expands to:

$$\begin{aligned} \vec{r}'(\varphi) \times \vec{r}''(\varphi) = & \left(\frac{dy}{d\varphi} \frac{d^2z}{d\varphi^2} - \frac{dz}{d\varphi} \frac{d^2y}{d\varphi^2} \right) \cdot \vec{i} \\ & - \left(\frac{dx}{d\varphi} \frac{d^2z}{d\varphi^2} - \frac{dz}{d\varphi} \frac{d^2x}{d\varphi^2} \right) \cdot \vec{j} \\ & + \left(\frac{dx}{d\varphi} \frac{d^2y}{d\varphi^2} - \frac{dy}{d\varphi} \frac{d^2x}{d\varphi^2} \right) \cdot \vec{k} \end{aligned} \tag{10}$$

Further, the cube of the position-vector norm in the denominator of (6) is given by:

$$\left\| \vec{r}'(\varphi) \right\|^3 = \left[\sqrt{\left(\frac{dx}{d\varphi} \right)^2 + \left(\frac{dy}{d\varphi} \right)^2 + \left(\frac{dz}{d\varphi} \right)^2} \right]^3 \tag{11}$$

The actual expression of the Toroidal/Helical orbit-curvature is then obtained by substituting the first and second derivatives of the Cartesian coordinates $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ in the expressions (7), (8), and (11), and simplified to obtain:

$$\kappa(\varphi) = \frac{\sqrt{\frac{1}{8n^2}\left[k_0 + k_1 \cos\left(\frac{n-1}{n}\varphi\right) + k_2 \cos\left(2\frac{n-1}{n}\varphi\right) + k_3 \cos\left(3\frac{n-1}{n}\varphi\right) + k_4 \cos\left(4\frac{n-1}{n}\varphi\right)\right]}}{\left[\sqrt{(1 + c\cos\theta)^2 + c^2\left(\frac{n-1}{n}\right)^2}\right]^3} \tag{12}$$

where the five $\boldsymbol{K}_i$ coefficients are functions of the torus aspect-ratio $\boldsymbol{c}$, and of the number of turns $\boldsymbol{n}$:

$$k_0 = \frac{1}{8n^6}\left\{8n^6 + 4c^2n^2\{1 + 2(n-1)n[2 + 3(n-1)n]\} + c^4(n-1)^2\{8 + n\{n[57 + 10n(2n-5)]\}\}\right\} \tag{13}$$

$$k_1 = \frac{1}{2n^4}c\left\{3c^2(n-1)^2(1 + 2(n-1)n) + 2n^2[1 + n(3n-2)]\right\} \tag{14}$$

$$k_2 = \frac{1}{2n^4}c^2\left\{\{2n[2 + n(2n-3)] - 1\} + (n-1)^2[2 + n(3n-4)]\right\} \tag{15}$$

$$k_3 = \frac{1}{2n^4}c^3(n-1)^2[1 + 2n(n-1)] \tag{16}$$

$$k_4 = \frac{1}{8n^4}c^4(n-1)^2(2n-1) \tag{17}$$

while the electron-orbit *curvature-radius* $\boldsymbol{\rho}$ is quite obviously expressed by $\boldsymbol{\rho(\theta) = 1/\kappa(\theta)}$ and is a periodic function of the *wrapping-angle* $\boldsymbol{\theta}$, in the $\mathbf{0} \leq \boldsymbol{\theta} \leq (\mathbf{n-1})2\boldsymbol{\pi}$ drange (Figure 3), with period $\mathbf{0} \leq \boldsymbol{\theta} \leq 2\boldsymbol{\pi}$.

Figure 1 shows a 3D display of a Toroidal/Helical orbit, with $\boldsymbol{n} = 9$ and $\boldsymbol{c} = 0.2$, and includes the locus of its moving curvature-center (the orbit *"evolute" !*), while Figure 2 shows the corresponding projection on the X–Y plane. Finally, Figure 3 shows one period of the orbit curvature-radius for $\mathbf{0} \leq \boldsymbol{\theta} \leq 2\boldsymbol{\pi}$, normalized to $\boldsymbol{R} = 1$. The closed-form exact expressions of the Toroidal/Helical orbit-curvature, and curvature-radius provide the possibility of computing the value of the continuously-evolving dipole magnetic-field $\boldsymbol{B}$, required to guide an electron-beam of known given energy $\boldsymbol{E}$, expressed in Mev, along the given orbit. Preliminary numerical computations with $\boldsymbol{n} = 9$ and $\boldsymbol{c} = 0.2$ have shown the maximum and minimum values of the dipole magnetic-field to be $\boldsymbol{B}_{MAX}$=4274.66 Gauss, and respectively $\boldsymbol{B}_{MIN}$=2578.4 Gauss, for an electron-energy $\boldsymbol{E}$=150 Mev, and $\boldsymbol{B}_{MAX}$=1434.489 Gauss, and respectively $\boldsymbol{B}_{MIN}$=865.258 Gauss, for an electron-energy $\boldsymbol{E}$=50 Mev. The space-orientation of such dipole field would, however, necessarily need to be also continuously evolving, following the continuous evolution of the Toroidal/Helical orbit *torsion* . The general expression of the orbit-torsion is given by [11]:

$$\tau(\varphi) = \frac{\vec{r'}(\varphi) \times \vec{r''}(\varphi) \cdot \vec{r'''}(\varphi)}{\left\|\vec{r'}(\varphi) \times \vec{r''}(\varphi)\right\|^2}$$

The corresponding closed-form, exact expression for Toroida/Helical orbits will be given in a following report.

5. Conclusions

We proposed closed-from, exact expressions of the Toroidal/Helical orbit-curvature and curvature-radius by first computing the first and second derivatives of the x, y and z components of the orbit position-vector r as function of the wrapping-angle $\boldsymbol{\theta}$, and by then substituting those derivatives in the general expression of the curvature of a parametric space-curve in three dimensions. The closed-form exact expressions provide of the possibility of computing the value of the continuously-evolving dipole magnetic-field $\boldsymbol{B}$, required to guide an electron-beam of known given energy $\boldsymbol{E}$, expressed in Mev , along the given orbit. Also, preliminary numerical computations were performed.

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