# Direct and Accurate FDTD Modeling of Dispersive Media Using a Fourth-order Rational Conductivity Function

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To analyze lossy, frequency dependent media over a wide RF bandwidth with FDTD, it is important to capture the wave velocity and attenuation with a simple, efficient model. Using a single pole rational function of the Z-transform variable ($Z = e^{jw\Delta t}$) to model media conductivity along with constant real dielectric constant, it is possible to generate a supplemental discretized time domain equation which closely matches measured values across more than a decade of frequency. The agreement between measured and modeled propagation constant and decay rate for more than 50 materials are often to within 5%. This formulation avoids memory-intensive convolution operations and is at least as accurate as Debye models.

In the FDTD formulation, with electric field sampled at integer time steps $\overline{E}^n$, and magnetic field sampled at half-integer steps $\overline{H}^{n-\frac{1}{2}}$, Ampere's Law presents a difficulty with the current term, which is computed using electric field but which must be available at the magnetic field time instant. This is accomplished by choosing an average current value between adjacent time steps $\overline{J}^{n-\frac{1}{2}} = \frac{\sigma}{2}(\overline{E}^n + \overline{E}^{n-1})$. The central finite differences used in FDTD are second order accurate, while the averaging over adjacent time steps is only first order accurate. A more precise solution is available using the Z-transform formulation of Ampere's Law:

$$\nabla \times \overline{H}(Z) = \frac{1 - Z^{-1}}{\Delta t} \in \overline{E}(Z) + Z^{-\frac{1}{2}}\sigma(Z)\overline{E}(Z) \tag{1}$$

with the understanding that $\overline{E}(Z)$ and $\overline{H}(Z)$ transform to integer and half-integer time samples. The Z-transformed current $\overline{J}(Z) = \sigma(Z)\overline{E}(Z)$, but only when the current values are sampled at the same time instances as the electric field. To keep the time sample alignment of current in synchronism with magnetic field, the last term on the right hand side of Eq. 1 transforms to $\overline{J}^{n-\frac{1}{2}}$. Keeping the finite difference equation form of the constitutive relation relating shifted current to electric field, the new rational function representation of conductivity is:

$$Z^{-\frac{1}{2}}\sigma(Z) = \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2} + b_3 Z^{-3}}{1 + a_1 Z^{-1}} \tag{2}$$

With this choice, the entire right hand side of Eq. 1 remains a rational function of integer powers of Z, and thus it can be readily converted to finite difference form. The additional term $b_3 Z^{-3}$ in Eq. 2 becomes necessary to ensure three point fitting, with proper curvature, of the conductivity function to measured data. The real part of conductivity, based on Eq. 2, is:

$$Re\{\sigma(Z)\} = \frac{(b_0 + b_1 + a_1(b_1 + b_2))\cos\omega\Delta t/2 + (b_2 + a_1(b_0 + b_3))\cos 3\omega\Delta t/2 + b_3 \cos 5\omega\Delta t/2}{1 + 2a_1 \cos\omega\Delta t + a_1^2} \tag{3}$$

with five parameters $b_0, b_1, b_2$, and $b_3$ to eb determined from matching to measured data. The parameter $a_1$ is adjusted to satisfy special von Neumann stability conditions requiring that all zeros of the stability equation be within the unit circle for a particular grid spacing interval.

*This work is supported by CenSSIS, the Center for Subsurface Sensing and Imaging Systems under the ERC Program of the NSF (Award number EEC-9986821)