A Subspace-based Robust Adaptive Capon Beamforming

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Abstract—Adaptive beamforming suffers from performance degradation in the presence of mismatch between the actual and presumed array steering vector of the desired signal. This idea enlightens us, so we propose a subspace approach to adaptive beamforming that is robust to array errors based on minimizing MUSIC output power. The proposed method involves two steps, the first step is to estimate the actual steering vector of the desired signal based on subspace technique, and the second is to obtain optimal weight by utilizing the estimated steering vector. Our method belongs to the class of diagonal loading, but the optimal amount of diagonal loading level can be calculated precisely based on the uncertainty set of the steering vector. To obtain noise subspace needs eigen-decomposition that has a heavy computation load and knows the number of signals *a priori*. In order to overcome this drawback we utilized the POR (Power of R) technique that can obtain noise subspace without eigen-decomposition and the number of signals *a priori*. It is very interesting that *Li Jian's* method is a special case where $m = 1$, and the proposed subspace approach is the case where $m \to \infty$, so we obtained a uniform framework based on POR technique. This is also an explanation why the performance of the proposed subspace approach excels that of *Li Jian's* method. The excellent performance of our algorithm has been demonstrated via a number of computer simulations.

1. Introduction

Array signal processing has wide applications in radar, communications, sonar, acoustics, seismology, and medicine. One of the important tasks of array processing is beamforming. The standard beamformers include the delay-and-sum approach, which is known to suffer from poor resolution and high sidelobe problems. The Capon beamformer adaptively selects the weight vectors to minimize the array output power subject to the linear constraint that the signal of interest (SOI) does not suffer from any distortion [1]. The Capon beamformer has better resolution and interference rejection capability than the standard beamformer, provided that the array steering vector corresponding to the SOI is accurately known. In practice, the knowledge of the SOI steering vector may be imprecise, the case due to differences between the presumed signal steering vector and the actual signal steering vector. When this happens, the Capon beamforming may suppress the SOI as an interference, which result in array performance drastically reduced, especially array output signal-to-interference-plus-noise ratio (SINR) [4].

In the past three decades many approaches have been proposed to improve the robustness of the Capon beamforming. Additional linear constraints, including point and derivative constraints, have been imposed to improve the robustness of the Capon beamforming [2, 3]. However, for every additional linear constraints imposed, the beamformer loses one degree of freedom (DOF) for interference suppression. Moreover, these constraints are not explicitly related to the uncertainty of the array steering vector. Diagonal loading (including its extended versions) has been a popular approach to improve the robustness of the Capon beamformer [4]. However, for most of the diagonal loading methods, determining the diagonal loading remains an open problem. Recently there are some methods been proposed (for examples, [5–7] and reference therein) to this point.

Mismatch between the presumed steering vector of the SOI and the actual one result in drastically reduced array SINR, therefore if we can estimate actual steering vector of the SOI, robustness of the array will be improved. In this paper, from the point of view of the subspace we propose a novel robust Capon beamformer, which involves two steps, the first step is to estimate actual steering vector of SOI, and the second is to calculate optimal weight by Capon method. The rest of this paper is organized as follows. Section 2 contains background material. In section 3, the robust Capon beamformer is developed. Computer simulation results illustrating the performance of the robust Capon beamformer are presented in Section 4. Finally, Section 5 contains the conclusions.

2. Background

2.1. Signal Model

We consider the standard narrowband beamforming model in which a set of M narrowband plane wave signals, impinge on an array of N sensors with half wavelength spacing, where $M < N$. The $N \times 1$ vector of received signals is given by

$$\mathbf{x}(t_k) = \sum_{m=0}^{M-1} \mathbf{a}(\theta_m) s_m(t_k) + \mathbf{n}(t_k), \quad k = 1, 2, \ldots, L \tag{1}$$

where $s_m(t_k), m = 0, \ldots, M-1;\ k = 1, 2, \ldots, L$ are the source signals snapshots,

$$\mathbf{a}(\theta_m) = [1, e^{-j\pi \sin\theta_m}, \ldots, e^{-j\pi(N-1)\sin\theta_m}]^T$$

is the steering vector in the direction θ_m, and $\mathbf{n}(t_k)$, $k = 1, 2, \ldots, L$ are the vectors containing additive white noise samples, L is the number of the snapshots. Also, in this paper, the sources and noise are assumed to be statistically uncorrelated.

We assume that one of the signals is the desired signal, say $s_0(t)$, and treat the remaining signals as interferences. Since $s_0(t)$ is uncorrelated with the noise and interferences, the data covariance matrix has the form,

$$\mathbf{R} = \sigma_0^2 \mathbf{a}(\theta_0)\mathbf{a}^H(\theta_0) + \sum_{k=1}^{M-1} \sigma_k^2 \mathbf{a}(\theta_k)\mathbf{a}^H(\theta_k) + \mathbf{R}_n \triangleq \mathbf{R}_s + \mathbf{R}_{i+n} \tag{2}$$

where $\mathbf{R}_s = \sigma_0^2 \mathbf{a}(\theta_0)\mathbf{a}^H(\theta_0)$, $\sigma_i^2 = E\{|s_i(t_k)|^2\}$ is the power of ith signal, and $\mathbf{R}_{i+n}$ is the interference plus noise covariance matrix. In practice, the covariance matrix $\mathbf{R}$ is estimated by

$$\hat{\mathbf{R}} = \frac{1}{L} \sum_{n=1}^{L} \mathbf{x}_n \mathbf{x}_n^H \tag{3}$$

where all received signals have zero means and L samples are independent.

2.2. Capon Beamforming

The Capon beamforming is as follows.

Determine the $N \times 1$ vector $\mathbf{w}_0$ that is the solution to the following linearly constrained quadratic minimization problem,

$$\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R} \mathbf{w} \quad s.t. \mathbf{w}^H \bar{\mathbf{a}}(\theta_0) = 1 \tag{4}$$

where $\bar{\mathbf{a}}(\theta_0)$ is presumed steering vector of the desired signal.

Appling Lagrange multiplier method results in the following solution,

$$\mathbf{w}_0 = \frac{\mathbf{R}^{-1}\bar{\mathbf{a}}(\theta_0)}{\bar{\mathbf{a}}^H(\theta_0)\mathbf{R}^{-1}\bar{\mathbf{a}}(\theta_0)} \tag{5}$$

The array mean output power p_0 is

$$p_0 = \frac{1}{\bar{\mathbf{a}}^H(\theta_0)\mathbf{R}^{-1}\bar{\mathbf{a}}(\theta_0)} \tag{6}$$

The Capon beamformer has better resolution and much better interference rejection capability than the standard beamformer, provided that the presumed array steering vector of the SOI match actual array steering vector precisely. In practice, the exact steering vector of the SOI is unavailable or its measure/estimation is imprecise, therefore, we only use the presumed $\bar{\mathbf{a}}(\theta_0)$ instead of the actual $\mathbf{a}(\theta_0)$ in the Capon beamformer, and the mismatch between the exact steering vector and the presumed one may drastically degrade the performance of the Capon beamformer.

The array output SINR can be written as,

$$SINR = \frac{E[|\mathbf{w}_0^H \mathbf{s}_0(t)|^2]}{\mathbf{w}_0^H \mathbf{R}_{i+n} \mathbf{w}_0} = \frac{\sigma_0^2 |\mathbf{w}_0^H \mathbf{a}(\theta_0)|^2}{\mathbf{w}_0^H \left(\sum_{k=1}^{M-1} \sigma_k^2 \mathbf{a}(\theta_k)\mathbf{a}^H(\theta_k) + \mathbf{R}_n \right) \mathbf{w}_0} \tag{7}$$

where $\sigma_0^2 = E(|s_0(t)|)$. Inserting (5) into (7) yields,

$$SINR = \sigma_0^2 \frac{\left|\bar{\mathbf{a}}^H(\theta_0)\mathbf{R}_{i+n}^{-1}\mathbf{a}(\theta_0)\right|^2}{\bar{\mathbf{a}}^H(\theta_0)\mathbf{R}_{i+n}^{-1}\bar{\mathbf{a}}(\theta_0)} \tag{8}$$

where $\mathbf{a}(\theta_0)$ is the actual steering vector, then (8) can be rewritten as:

$$\begin{aligned} SINR &= \sigma_0^2 \mathbf{a}^H(\theta_0)\mathbf{R}_{i+n}^{-1}\mathbf{a}(\theta_0) \times \frac{\left|\mathbf{a}^H(\theta_0)\mathbf{R}_{i+n}^{-1}\bar{\mathbf{a}}(\theta_0)\right|^2}{(\mathbf{a}^H(\theta_0)\mathbf{R}_{i+n}^{-1}\mathbf{a}(\theta_0))(\bar{\mathbf{a}}^H(\theta_0)\mathbf{R}_{i+n}^{-1}\bar{\mathbf{a}}(\theta_0))} \\ &= SINR_m \cdot \cos^2(\mathbf{a}(\theta_0), \bar{\mathbf{a}}(\theta_0); \mathbf{R}_{i+n}^{-1}) \end{aligned} \tag{9}$$

where $SINR_m = \sigma_0^2 \mathbf{a}^H(\theta_0)\mathbf{R}_{i+n}^{-1}\mathbf{a}(\theta_0)$ and $\cos^2(\cdot)$ is defined as,

$$\cos^2(\mathbf{a}, \mathbf{b}; \mathbf{Z}) = \frac{\left|\mathbf{a}^H\mathbf{Z}\mathbf{b}\right|^2}{(\mathbf{a}^H\mathbf{Z}\mathbf{a})(\mathbf{b}^H\mathbf{Z}\mathbf{b})} \tag{10}$$

Clearly, $0 \le \cos^2(\mathbf{a}, \mathbf{b}; \mathbf{Z}) \le 1$. Therefore, array output SINR is reduced due to mismatch between the presumed steering vector of the SOI and its true value.

In recent years, diagonal loading (DL) is a popular approach to improving the robustness of Capon beamformer to the mismatch above. In DL methods, the data covariance $\hat{\mathbf{R}}$ is replaced by $\hat{\mathbf{R}} + \gamma\mathbf{I}$, where γ is positive constant (see reference [4–6] for details). The DL method proposed in [4] is used in Section 4 for comparisons. In the following section, a novel robust beamforming is developed to alleviate the effects of the steering vector mismatch on the SINR performance of Capon beamformer.

3. Robust Capon Beamforming

The *robust beamforming problem* we will deal with in this paper can be briefly stated as follows: Extend the Capon beamformer so as to improve array output SINR even only an imprecise knowledge of steering vector $\mathbf{a}(\theta_0)$ is available. To simplify the notation, in what follows, we sometimes omit the argument θ of $\mathbf{a}(\theta)$ and $\bar{\mathbf{a}}(\theta)$. We assume that the only knowledge we have about $\mathbf{a}(\theta_0)$ is that it belongs to the following uncertainty [5]

$$[\mathbf{a}(\theta_0) - \bar{\mathbf{a}}]^H\mathbf{C}^{-1}[\mathbf{a}(\theta_0) - \bar{\mathbf{a}}] \le 1 \tag{11}$$

where $\mathbf{C}$ are given positive definite matrix.

As shown above, array performance loses will occur in the presence of mismatch between the presumed and actual steering vectors of the SOI. If we estimate the actual steering vector of the SOI as more precise as we can, then performance of the beamformer will be improved. The proposed robust Capon beamforming is based on this idea. From subspace theory we know that the actual steering vector of desired signal is orthogonal to noise subspace, our approach is based on the optimizing the projection of signal steering vector onto noise subspace. The steering vector is normed as $||\mathbf{a}||^2 = \mathbf{a}^H\mathbf{a} = N$. To derive our robust Capon beamformer, we use following constrained optimization

$$\begin{aligned} &\min_{\mathbf{a}} \mathbf{a}^H\mathbf{U}_n\mathbf{U}_n^H\mathbf{a} \\ &s.t. (\mathbf{a} - \bar{\mathbf{a}})^H C^{-1}(\mathbf{a} - \bar{\mathbf{a}}) \le 1 \\ &||\mathbf{a}||^2 = N \end{aligned} \tag{12}$$

where $\bar{\mathbf{a}}$ is known to us in advance, but has error (mismatch to the actual steering vector of the SOI). $\mathbf{U}_n$ is the noise subspace, which is obtained by the eigen-decomposition of $\hat{\mathbf{R}}$. To make up the noise subspace, we assume that the number M, of plane waves impinging on the array is known *a priori*. We use this assumption only for derivations and cancel it later. Note that we can improve the estimation accuracy of the actual steering vector of the SOI from (12), and then obtain optimal weight $\mathbf{w}_0$ by Capon method.

Without loss of generality, we will consider solving (12) for the case in which $\mathbf{C} = \varepsilon\mathbf{I}$, ($\varepsilon$ is user parameter), then, (12) becomes

$$\begin{aligned} &\min_{\mathbf{a}} \mathbf{a}^H\mathbf{U}_n\mathbf{U}_n^H\mathbf{a} \\ &s.t. ||\mathbf{a} - \bar{\mathbf{a}}||^2 \le \varepsilon \\ &||\mathbf{a}||^2 = N \end{aligned} \tag{13}$$

We use the *Lagrange multiplier methodology* again, which is based on the function

$$L(\mathbf{a}, \lambda, \mu) = \mathbf{a}^H\mathbf{U}_n\mathbf{U}_n^H\mathbf{a} + \mu(2N - \varepsilon - \bar{\mathbf{a}}^H\mathbf{a} - \mathbf{a}^H\bar{\mathbf{a}}) + \lambda(\mathbf{a}^H\mathbf{a} - N) \tag{14}$$

where $\mu \ge 0$, $\lambda \ge 0$ are the *Lagrange multiplier*.

Hence, the unconstrained minimization of (14) for fixed μ, λ, is given by

$$\frac{\delta L(\mathbf{a}, \mu, \lambda)}{\delta \mathbf{a}} = 2\mathbf{U}_n\mathbf{U}_n^H\mathbf{a} - 2\mu\bar{\mathbf{a}} + 2\lambda\mathbf{a} = 0 \tag{15}$$

Clearly, the optimal solution of $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \mu(\mathbf{U}_n\mathbf{U}_n^H + \lambda\mathbf{I})^{-1}\bar{\mathbf{a}} \tag{16}$$

Inserting $\hat{\mathbf{a}}$ into (14), minimizing $L(\mathbf{a}, \lambda, \mu)$ with respect to μ gives

$$\frac{\delta L(\hat{\mathbf{a}}, \mu, \lambda)}{\delta \mu} = 2N - \varepsilon - \bar{\mathbf{a}}^H \hat{\mathbf{a}} - \hat{\mathbf{a}}^H \bar{\mathbf{a}} = 0 \tag{17}$$

Then, we obtain

$$\hat{\mu} = \frac{2N - \varepsilon}{2\bar{\mathbf{a}}^H (\mathbf{U}_n \mathbf{U}_n^H + \lambda \mathbf{I})^{-1} \bar{\mathbf{a}}} \tag{18}$$

Inserting $\hat{\mu}$ into (14), minimizing Lagrange function with respect to λ yields

$$\frac{\delta L(\hat{\mathbf{a}}, \hat{\mu}, \lambda)}{\delta \lambda} = \hat{\mathbf{a}}^H \hat{\mathbf{a}} - N = 0 \tag{19}$$

and the following equation can be derived,

$$\frac{\bar{\mathbf{a}}^H (\mathbf{U}_n \mathbf{U}_n^H + \hat{\lambda} \mathbf{I})^{-2} \bar{\mathbf{a}}}{[\bar{\mathbf{a}}^H (\mathbf{U}_n \mathbf{U}_n^H + \hat{\lambda} \mathbf{I})^{-1} \bar{\mathbf{a}}]^2} = \frac{N}{(N - \frac{\varepsilon}{2})^2} \tag{20}$$

Then, the solution of $\hat{\lambda}$ can be obtained by some simple manipulations.

Substituting (18) into (16) yields

$$\hat{\mathbf{a}} = (N - \frac{\varepsilon}{2}) \frac{(\mathbf{U}_n \mathbf{U}_n^H + \hat{\lambda} \mathbf{I})^{-1} \bar{\mathbf{a}}}{\bar{\mathbf{a}}^H (\mathbf{U}_n \mathbf{U}_n^H + \hat{\lambda} \mathbf{I})^{-1} \bar{\mathbf{a}}} \tag{21}$$

To summarize, the proposed robust Capon beamforming consists of following steps.

The algorithm:

Step 1: Calculate data covariance matrix, i.e.,

$$\hat{\mathbf{R}} = \frac{1}{L} \sum_{n=1}^{L} \mathbf{x}_n \mathbf{x}_n^H$$

Step 2: Compute the eigen-decomposition of $\hat{\mathbf{R}}$ and obtain the noise subspace $\mathbf{U}_n$.

Step 3: Solve $\hat{\lambda}$ in (20).

Step 4: Use the $\hat{\lambda}$ in Step 3 to calculate

$$\hat{\mathbf{a}} = (N - \frac{\varepsilon}{2}) \frac{(\mathbf{U}_n \mathbf{U}_n^H + \hat{\lambda} \mathbf{I})^{-1} \bar{\mathbf{a}}}{\bar{\mathbf{a}}^H (\mathbf{U}_n \mathbf{U}_n^H + \hat{\lambda} \mathbf{I})^{-1} \bar{\mathbf{a}}} \tag{22}$$

Step 5: Compute optimal weight by Capon method, i.e.,

$$\mathbf{w}_0 = \alpha \hat{\mathbf{R}}^{-1} \hat{\mathbf{a}}, \;\; \alpha = \frac{1}{\hat{\mathbf{a}}^H \hat{\mathbf{R}}^{-1} \hat{\mathbf{a}}} \tag{23}$$

The proposed robust beamforming belongs to the class of diagonal loading, but the optimal amount of diagonal loading level can be precisely calculated based on the uncertainty set of the steering vector. In the Section 4 computer simulation results demonstrate excellent performance of the proposed algorithm.

In order to avoid eigen-decomposition and knowing the number of signals *a priori*, we use the POR approach to obtain noise subspace. In [8], $\mathbf{R}$ is decomposed by EVD as

$$\mathbf{R} = [\mathbf{U}_s \; \mathbf{U}_n] \begin{bmatrix} \Lambda_s + \sigma_v^2 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma_v^2 \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{U}_s^H \\ \mathbf{U}_n^H \end{bmatrix} \tag{24}$$

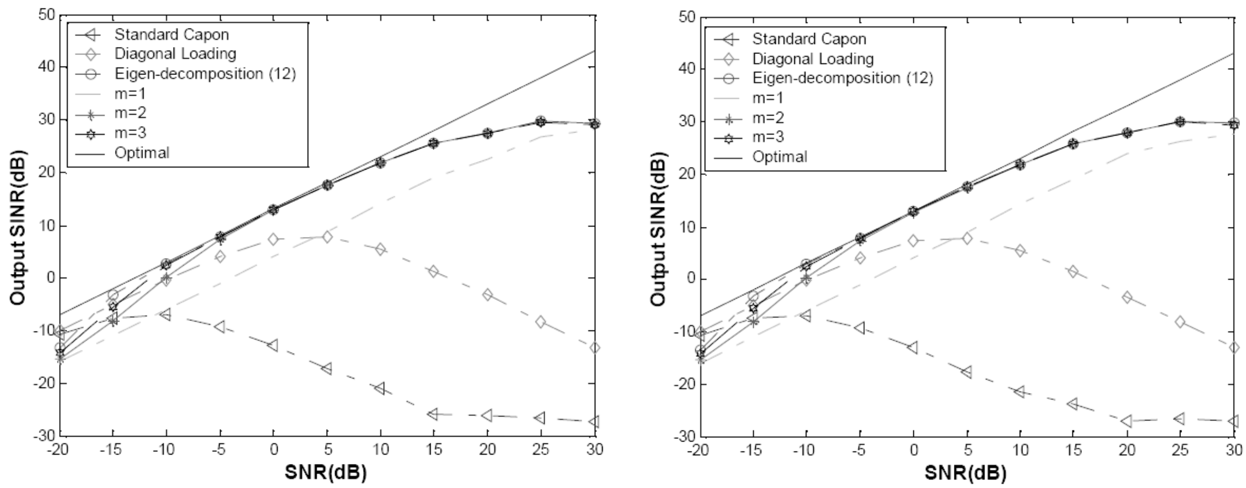
where $\Lambda_s = diag\{\delta_1^2, \ldots, \delta_M^2\}$, $\mathbf{U}_s$ denotes the signal subspace. It approximates the noise subspace of $\mathbf{R}$ based on $\mathbf{R}^{-m}$ (m is a positive integer). Accordingly

$$\sigma_v^{2m} \mathbf{R}^{-m} = \mathbf{U}_n \mathbf{U}_n^H + \mathbf{U}_s diag\Big\{ \Big(\frac{\sigma_v^2}{\delta_i^2 + \sigma_v^2} \Big)^m \Big\} \mathbf{U}_s^H \tag{25}$$

Clearly, $(\sigma_v^2/(\delta_i^2 + \sigma_v^2))^m$ is less than 1 and converge to zero for sufficiently large m. Theoretically, $\lim_{m\to\infty} \sigma_v^{2m} \mathbf{R}^{-m} = \mathbf{U}_n \mathbf{U}_n^H$. As result, we modify the criterion (12) and consider the following POR cost function

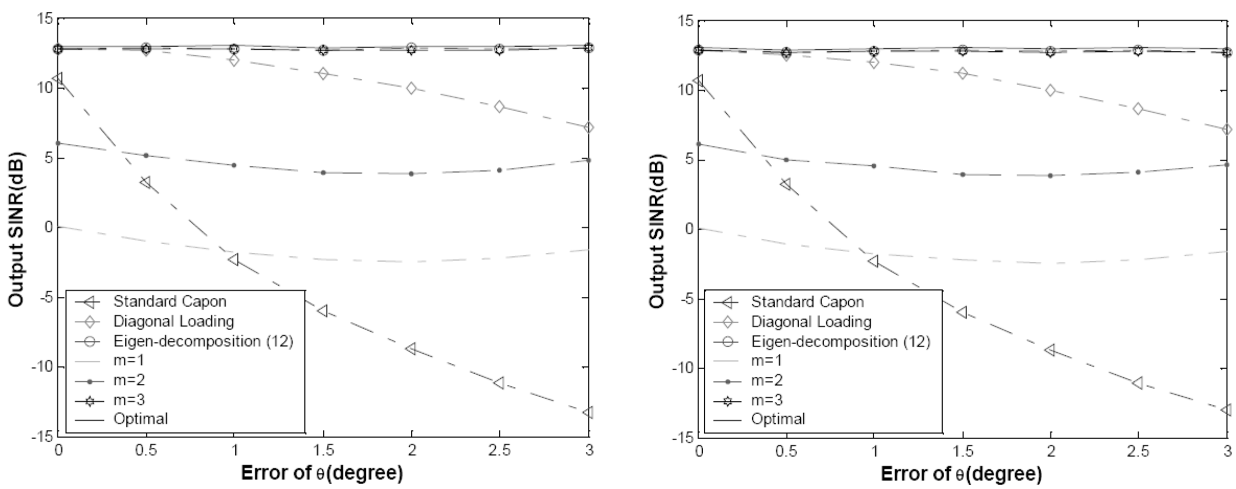
$$\begin{aligned} &\min_{\mathbf{a}} \mathbf{a}^H \hat{\mathbf{R}}^{-m} \mathbf{a} \\ &s.t. ||\mathbf{a} - \bar{\mathbf{a}}||^2 \leq \varepsilon \\ &||\mathbf{a}||^2 = N \end{aligned} \tag{26}$$

By contrast, the (26) avoids estimating that dimension directly. Moreover, as $m \to \infty$, the proposed the POR beamforming method in (26) converges to the subspace one in (12), and it can be shown that the performance of the POR method for finite m will converge to the subspace one through computer simulation. We compared our method with previous one in [6], where $m = 1$ in the section 4.



(a) (b)

Figure 1: Output SINR versus different SNR, pointing errors $\Delta = 3°$, for (a) $\varepsilon = 0.7$, for (b) $\varepsilon = 7$.



(a) (b)

Figure 2: The Output SINR versus pointing errors for (a) $\varepsilon = 0.7$, for (b) $\varepsilon = 7$.

4. Computer Results

Our main motivation of simulation is to demonstrate the performance in the presence of some errors in the steering vector. In all of the examples considered below, we assume a uniform linear array (ULA) with $N = 20$ sensors and half-wavelength spacing is used. The sources emitted mutual independent narrowband waveforms. All the results are achieved via 50 Monte Carlo trials.

In the first example, we consider the effect of the pointing error of the SOI on array output SINR. The exact direction of arrival of SOI is θ_0, of which assumed value is $\theta_0 + \Delta$, i.e., $\bar{\mathbf{a}}(\theta_0) = \mathbf{a}(\theta_0 + \Delta)$. We assume that $\mathbf{a}(\theta_0)$ belongs to the uncertainty set

$$||\mathbf{a}(\theta_0) - \bar{\mathbf{a}}(\theta_0)||^2 \leq \varepsilon \tag{27}$$

where ε is a user parameter. Let $\varepsilon_0 = ||\mathbf{a}(\theta_0) - \bar{\mathbf{a}}(\theta_0)||^2$. Then, choosing $\varepsilon = \varepsilon_0$. However, since Δ is unknown in practice, the ε we choose may be greater or less than ε_0. To show that the choice of ε is not a critical issue for our algorithm, we will present simulation results with several values of ε in equation (21). In this example, the directions of the SOI and an interference source are $\theta_0 = 30°$, $\theta_1 = -30°$, respectively. The assumed direction of the SOI is $\theta_0 + \Delta = 33°$, which results exact $\varepsilon_0 = 5.7750$. The interference-to-noise ratio (INR) is 40 dB.

Figure 1 plots array output SINR versus the SNR of the SOI when the number of snapshots is set to be $L = 100$. It is observed that the proposed algorithm (12) performs better than other two algorithms at all input SNR. Also, since the error in steering vector of SOI is relatively large and cannot be negligible, the standard Capon beamformer and its diagonal loading version suffer from severe performance degradation when SNR increases. However, the proposed beamformer has SINR loss of 5 dB when SNR = 20 dB. The proposed the POR method for different m over various input SNRs is also illustrated in Figure 1. Obviously, the Output SINR for $m = 2$ and $m = 3$ all converge to subspace approach (12), the counterpart for $m = 1$ [6] has the large output SINR loss.

Figure 2 shows the array output SINR curve versus the pointing errors, in which SNR = 0 dB, INR = 20 dB, $L = 100$. In this figure, the excellent performance achieved by the proposed algorithm is observed, which shows the robustness to the pointing errors. It is noted that, similar to other robust approaches, our method will worsen if there is/are strong interference spatially closed to the SOI. The reason is that for a given uncertainty region (11), the solution of $\mathbf{a}$ in optimization (12) is converge to the strong interference source. Also, it can be seen that the Output SINR of the proposed POR method increases as m increases, with $m = 3$ has same performance with subspace one (12).

5. Conclusion

In this paper, we discuss the performance degradation due to the presence of steering vector uncertainty of the SOI, such as, direction of arrival estimation error, finite number of snapshots, and array response error, etc. A robust Capon beamformer is developed by utilizing the orthogonality between signal and noise subspace. A more accurate estimate of the actual steering vector of the SOI is obtained via constrained optimization, by which the optimal weight is computed according to Capon beamforming. We have shown that the proposed algorithm belongs to the class of diagonal loading approaches, and the optimal amounts of diagonal loading can be precisely calculated. In order to avoid eigen-decomposition and knowing the number of signals *a priori*, we have proposed a POR-based robust beamforming scheme. It significantly outperforms the method proposed in [6] and converge to the subspace one (12). The excellent performance of our algorithm has been demonstrated via a number of computer simulations.

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