The Spectral Expansion on the Entire Real Line of Green's Function for a Three-layer Medium in the Fundamental Functions of a Nonself-adjoint Sturm-Liouville Operator

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Abstract—We obtain a new representation for Green's function in the space R^2 of the Helmholtz equation with the coefficient representing a complex-valued piecewise constant function. We set that the coefficient in the equation depends on the one variable and represents three complex constants.

This representation is the expansion of Green's function in the fundamental functions which are bounded on the entire real line R^1 solutions of the ordinary Sturm-Liouville equations with complex coefficients.

The spectrum consists of two half-lines parallel to the real axis on the complex plane, issuing from the points characterized by the coefficient in the equation on semi-infinite intervals, and going in the positive direction of the real axis.

1. Introduction

In the present paper, we obtain a new representation ([1, 2]) for the solution of a problem for a three-layer medium similar to the problem on a dipole in the space containg a plane interface between two media characterized by constant wave numbers $k_i, i = 1, 2$. The latter problem was considered in [3].

The new representation follows from the representation of Green's function in the form of a Fourier integral obtained by reducing an integral on the complex plane of the spectral parameter to integrals over the edges of cuts passing through the points k_i, $i = 0, 2 (k_i = const)$ characterized by the coefficient $k(z)$ of the equation outside a finite interval $[0, H]$ where $k(z) = k_l = const(l = 1)$. The function $k(z)$ represents a complex-valued piecewise constant function of a variable z on the entire real line R^1.

2. Formulations and Equations

Green's function $u(z, x)$ satisfies the Helmholz equation

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} + k^2(z)u = -2\delta(z - z')\delta(x - x') \tag{1}$$

with the delta function on the right-hand side and with either the radiation condition $u \to 0$ as $r = \sqrt{z^2 + x^2} \to \infty$ [if $Im\, k(z) \neq 0$] or the radiation condition following from the limiting absorption principle [if $Imk(z) = 0$]. The function u is a bounded function in R^2 with the exception of the source position $M'(z', x')$ where it has a logarithmic singularity. At the points $z = 0, z = H$ of discontinuity of the function $k(z)$, the function $u(z, x)$ satisfies the matching conditions for the function and its normal derivative on the boundary. We consider Eq. (1)under the assumption that $z, z', x, x' \in R^1$, where R^1 is the real axis. We set

$$k(z) = \begin{cases} k_0 = const, z < 0, \\ k_1 = const, 0 < z < H, \\ k_2 = const, z > H. \end{cases}$$

where the k_l are complex constants, $k_l^2 = \varepsilon_l + j\sigma_l$, $\varepsilon_l \in R^1$, $\varepsilon_0 = \varepsilon_2, \sigma_l \geq 0$, $\sigma_0 < \sigma_1 < \sigma_2$ or $\sigma_0 > \sigma_1 > \sigma_2$, $l = 0, 1, 2$, and j is the imaginary unit.

To find the solution, we consider the Fourier expansion of u with respect to the variable x [of which the cofficient in Eq. (1) is independent]:

$$u = \lim_{\Lambda \to \infty} u_\Lambda, \qquad u_\Lambda = \frac{1}{\pi} \int_{-\Lambda}^{\Lambda} e^{j\alpha(x - x')} g(z, z'; \alpha)\, d\alpha. \tag{2}$$

The function g can be found from the equation

$$lg + \alpha^2 g = \delta(z - z'), \qquad z, z' \in R^1, \tag{3}$$

where l is the differential operator

$$l\psi = -d^2\psi/dz^2 - k^2(z)\psi,$$

and has the form

$$g(z,z';\alpha)=\frac{\psi(z_>,\alpha)\varphi(z_<,\alpha)}{w(\alpha)}. \tag{4}$$

Here $z_>=max(z,z')$, $z_<=min(z,z')$, and $w(\alpha)$ is the Wronskian of the linearly independent solutions ψ and φ of the homogeneous Eq. (3). The function g is bounded for $z,z'\in R^1$ and satisfies the matching condition for the function and its derivative dg/dz at the points $z=0$ and $z=H$.

Taking into account the representation (4) of the function g via the functions φ,ψ, and w, we find that the function g has the ramification points $\pm k_i$ in the complex plan α ([4] p. 23–25, Vol. 2 of the Russian translation).

The cuts corresponding to the ramification points k_i go along the lines

$$\alpha=\sqrt{-\mu^2+k_i^2}=j\sqrt{\mu^2-k_i^2},\qquad \mu\in R^1,\ i=0,2.$$

We assume that $Re\sqrt{-\mu^2+k_i^2}\geq 0$, and $Re\sqrt{\mu^2-k_i^2}\geq 0$.

Consider the case in which $\varepsilon_0=\varepsilon_2$, $\sigma_2>\sigma_0$, $\sigma_0>0$, $\sigma_2>\sigma_1>\sigma_0$.

The Wronskian $w(\alpha)\neq 0$ on the entire complex α-plane [5, 6].

Consider the case $x-x'>0$.

Using the Cauchy theorem, we reduce the integral u_Λ given by (2) in the upper half of the complex α-plane to two integrals over the edges of the cuts passing through the points k_i and the integral I_{C_Λ} over the half-circle C_Λ of the radius Λ.

We have

$$u_\Lambda(z,x)=\sigma_\Lambda(z,u)+I_{C_\Lambda}, \tag{5}$$

where

$$\sigma_\Lambda(z,u)=\frac{j}{\pi}\sum_{i=0,2}\int_{-M_i(\Lambda)}^{M_i(\Lambda)}\mu\frac{e^{-\sqrt{\mu^2-k_i^2}(x-x')}}{\sqrt{\mu^2-k_i^2}}\frac{\psi_i(z_>,\mu)\varphi_i(z_<,\mu)}{w_i(\mu)}d\mu. \tag{6}$$

The function $M_i(\Lambda)$, which define the limits of integration in (6), depend on Λ, and $M_i(\Lambda)\sim\Lambda$ as $\Lambda\to\infty$. The quantity $M_i(\Lambda)>0$ occuring in (6) is the value of μ at which the right edge of the cut passing through the point k_i intersects the half-circle C_Λ.

The functions $\psi_i(z,\mu)$ and $\varphi_i(z,\mu)$ are linearly independent solutions of the equations

$$l\chi_i=(\mu^2-k_i^2)\chi_i,\ \mu\in R^1,\ i=0,2. \tag{7}$$

They are related to the functions $\psi(z,\alpha)$ and $\varphi(z,\alpha)$ [which are linearly independent solutions of the equation $(l+\alpha^2)\chi=0$] by the formula $\chi(z,\mu)=\chi(z,\alpha=j(\mu^2-k_i^2)^{1/2})$.

The Wronskian on the cuts passing through the ramification points k_i is given by the formula $w_i(\mu)=w(\alpha=j(\mu^2-k_i^2)^{1/2})$.

The integral I_{C_Λ} occuring in (5) has the form

$$I_{C_\Lambda}=\frac{1}{\pi}\int_{C_\Lambda}e^{j\alpha(x-x')}g(z,z';\alpha)d\alpha,$$

where C_Λ is the half-circle of radius Λ centered at the point $\alpha=0$ in the upper half-plane of the complex variable α.

We introduce the functions $\eta^l=\sqrt{\alpha^2-k_l^2}, l=0,1,2, Re\eta^l\geq 0$. The representations of the functions ψ_i, φ_i, and w_i can be derived from ψ, φ, and w with regard to the the fact that $Im\eta^l<0$ in the domain lying on the left of the hyperbola $\alpha_2=\sigma_l/(2\alpha_1)$ passing through the point k_l in the upper half-plane of the variable $\alpha=\alpha_1+j\alpha_2$; next, $Im\eta^l>0$ in the domain on the right of the hyperbola $\alpha_2=\sigma_l/(2\alpha_1)$ passing through the point k_l ([4] p. 30, Vol.2 of the Russian translation). The following condition is satisfied on the cuts drawn along the hyperbolas: $\mu>0$ on the right edge of the cut, and $\mu<0$ on the left edge of the cut passing trough the points k_i, i=0, 2.

We have $I_{C_\Lambda}\to 0$ as $\Lambda\to\infty$.

Using the functions $u_0=\psi_0$ and $u_2=\varphi_2$, we can rewrite the function (6) as

$$\sigma_\Lambda(z,u)=\sum_{i=0,2}\int_0^{M_i(\Lambda)}\frac{e^{-\sqrt{\mu^2-k_i^2}(x-x')}}{\sqrt{\mu^2-k_i^2}}u_i(z,\mu)u_i(z',\mu)dp_i(\mu), \tag{8}$$

where $dp_0(\mu)=d\mu/a_0^0(\mu)b_0^0(\mu)2\pi$, $dp_2(\mu)=d\mu/a_2^2(\mu)b_2^2(\mu)2\pi$. a_i^i and b_i^i are coefficients connected with transmission and reflection coefficients of u_i.

In deriving (8), we represent the integral (6) as two integrals over the positive and negative semiaxis and make the change of variables $\mu'=-\mu$ in the integral over the negative semiaxis.

Passing to the limit in (5) as $\Lambda\to\infty$, we obtain the representation

$$u = \sum_{i=0,2} \int_0^\infty \frac{e^{-\sqrt{\mu^2-k_i^2}(x-x')}}{\sqrt{\mu^2-k_i^2}} u_i(z,\mu) u_i(z',\mu) dp_i(\mu). \tag{9}$$

which holds for $x - x' > 0$.

In a similar way, we consider the case $x - x' < 0$ by reducing the integral u_Λ with the use of the Cauchy theorem in the lower half-plane of the complex variable α to two integral over the edges of the cuts passing through the points $-k_i$.

Passing to the limit in (9) as $x \to x'$, we obtain relation (9)) with $x = x'$.

3. Conclusion

We have thereby obtained the definitive representation

$$u = \sum_{i=0,2} \int_0^\infty \frac{e^{-\sqrt{\mu^2-k_i^2}|x-x'|}}{\sqrt{\mu^2-k_i^2}} u_i(z,\mu) u_i(z',\mu) dp_i(\mu), \tag{10}$$

which is valid for $x, x'z, z' \in R^1$. This representation of Green's function u was obtained under the assumption that $\sigma_2 > \sigma_0 > 0\,(\sigma_2 > \sigma_1 > \sigma_0)$ The case $0 < \sigma_2 < \sigma_0\,(\sigma_2 < \sigma_1 < \sigma_0)$ can be treated in a similar way.

The representation (10) is the expansion of Green's function in the fundamental functions u_i, which are bounded on the entire real line R^1 solutions of the ordinary Sturm-Liouville Eq. (7) with complex coefficients. This expansion is characterized by the spectral measure, which is a diagonal matrix function with nonzero entries $p_i(\mu)$.

Equation (7) for the functions u_i indicates that the spectrum $\lambda = \mu^2 - k_i^2$, $\mu \in R^1$, consists of two half-lines parallel to the real axis on the complex λ-plane, issuing from the points $-k_i^2$, and going in the positive direction of the real axis.

Passing to the limit as $\sigma_2 \to \sigma_0$, we arrive the case $\sigma_0 = \sigma_2$.

If $\sigma_2 \to 0$ and $\sigma_0 \to 0$, then we obtain the limit case $\sigma_0 = \sigma_2 = 0$. Then the spectrum belongs to real axis, and the spectrum is double for $\lambda \geq -\varepsilon_0$. In this case, the lower bound of the spectrum is limited to the number $\lambda = -\varepsilon_0$. This case is an example of the expansion of a function of the class $L_p(R^1), p > 2$, in the fundamental functions of the Sturm-Liouville operator with a real coefficient $-k^2(z)$ satisfying the Kato condition ([7]).

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REFERENCES

1. Dmitriev, V. I. and E. G. Saltykov, "The new representation of Green's function of Sommerfeld's problem for a conductive homogeneous half-space," *Uspekhi Mat. Nauk* , Vol. 49, No. 4(298), 79–80, 1994.
2. Saltykov, E. G., "The spectral expansion on the entire real line of Green's function for two-layer medium in the eigenfunctions of a nonself-adjoint Sturm-Liouville operator," *Differents. Uravn.*, Vol. 38, No. 5, 687–691, 2002.
3. Sommerfeld, A., *Differentialgleichungen der Physik, Db. VI*, Leipzig, 1948. Translated under the title *Differentsialńye Uravneniya v Chastnykh Proizvodnykh Fiziki*, 358, Inostrannaya Literatura, Moscow, 1950.
4. Felsen, L. and N. Marcuvitz, *Radiation and Scattering of Waves*, Prentice-Hall, Englewood Cliffs, 1973. Translated under the title *Izluchenie i Rasseyanie Voln*, Moscow, 1978.
5. Saltykov, E. G., "Absence of poles in integrand of Green's function for a three-layer medium," *PIERS 2005 Progress In Electromagnetic Research Symposium, Abstracts*, 183, Hangzhou, China, 2005.
6. Dmitriev, V. I. and E. G. Saltykov, "Absense of poles in integrand of the Green's function for special model of three layers medium," *International conference "Differantial Equations and Related Topics"*, 111, Moscow, 2001.
7. Ilín, V. A., "Uniform equiconvergence of the spectral expansion corresponding to a self-adjoint extension of the Schrödinger operator with a uniformly locally integrable potential with the Fourier integral on the entire axis R," *Differents. Uravn.*, Vol. 31, No. 12, 1957–1967, 1995.