

Improved Mesh Conforming Boundaries for the TLM Numerical Method

I. J. G. Scott and D. de Cogan

University of East Anglia, UK

Abstract—The numerical simulation of wave propagation in bounded space using differential-based models generally encounter spatial discretisation problems when the boundaries of the computation space do not fall on exact multiples of the models discretisation. While the accuracy can be improved by refinement of the model, the computational load can increase exponentially, often making the problem unsolvable. There have been some previous attempts to achieve boundary conforming meshes for the TLM numerical method. This paper describes a novel approach which compares well with these methods with a significantly reduced computational load.

1. Introduction

While the Transmission Line Matrix (TLM) numerical method is becoming increasingly easy to utilise for a wider variety of electromagnetic problems, in part, due to the definition of perfectly matched loads (PMLs) [1–3], boundary conforming schemes often increase the computational load and/or complexity of the algorithm, making TLM difficult to implement for simulations of bounded regions that fluctuate rapidly along the periphery (Figure 1(a)). The dotted lines show the sections that will require extra analysis in order to meet with the boundary. While some TLM simulations can be approximated reasonably accurately with a stepped formulation (by extension/deletion of the dashed sections), in the case of most TLM simulations the errors introduced are unacceptable and refinement of the mesh is often performed, increasing both time and memory requirements. If the true distance, l_a in figure 1b could be incorporated within the simulation, the mesh would pertain much more closely to the true boundary/surface of the device being modelled.

The length of the transmission line in the TLM algorithm cannot simply be adjusted. As can be observed from Figure 1(b), a signal travelling along the full length ($\Delta x/2$, where Δx is the spatial discretisation in the model), will, after reflecting from the surface, appear back at the boundary adjacent node at time $(k+1)\Delta t$, the same signal travelling along the line of length l_A will appear back at the boundary adjacent node at a time less than $(k+1)\Delta t$, but greater than $k\Delta t$ (where k is the current iteration (discrete time step) and $k+1$ is the succeeding iteration). As propagating signals in TLM must all arrive in steps of the same discretised time (Δt), this cannot be modelled directly.

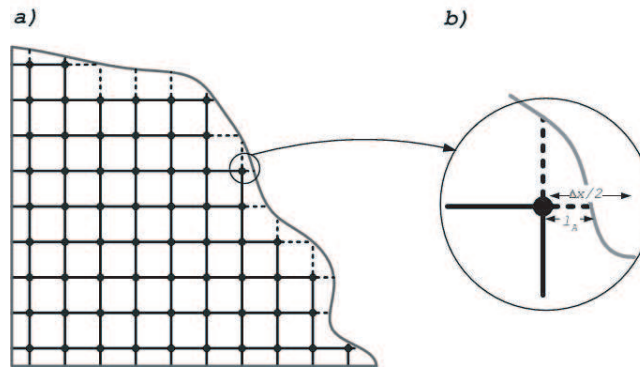


Figure 1: a) Cartesian mesh of non-stepped boundary, b) boundary adjacent node.

De Cogan and de Cogan [4] have adapted existing schemes to demonstrate the application of boundary-conforming finite difference schemes for the solution of the Laplace equation. In a uniformly bounded space where h is the length of the line segments (Figure 2) we use

$$T(x+h, y) + T(x, y+h) + T(x-h, y) + T(x, y-h) + 4T(x, y) = 0 \quad (1)$$

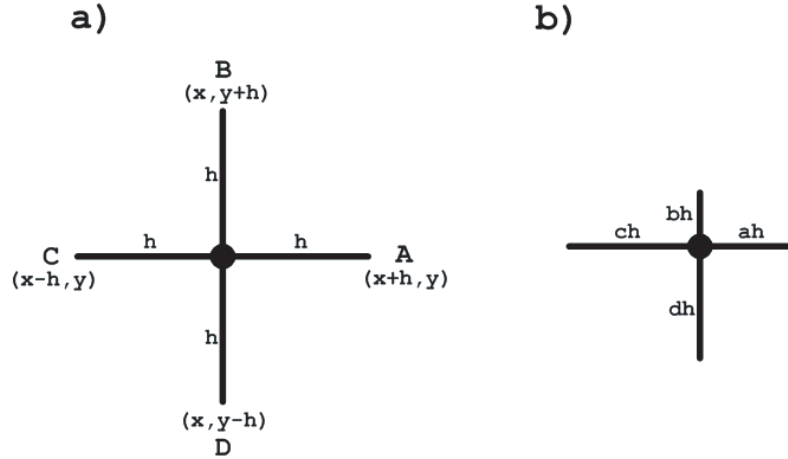


Figure 2: a) 2D FDTD visual representation, b) 2D FDTD irregular boundary visualisation.

In the situation where we take account of a node where we have unequal distances between nodes, given by ah , bh , ch and dh (h is the uniform line-length), the two-dimensional Laplacian becomes

$$\nabla^2 V = \frac{2}{h^2} \left[\frac{V_A}{a(a+c)} + \frac{V_B}{b(b+d)} + \frac{V_C}{c(c+a)} + \frac{V_D}{d(d+b)} - \frac{ac+bd}{abcd} V_0 \right] \quad (2)$$

where we consider the potentials of the four nodes, A, B, C, and D which surround the potential V_0 .

Using concepts developed in [5] it is easy to see how one might develop a boundary conforming implementation of the wave equation

$$\nabla^2 V = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} \quad (3)$$

Although FDTD is well established in electromagnetics and hybrid schemes involving finite difference are well known, there is not much evidence that such a mesh conforming scheme is widely used. Probably because time is implicitly discretised in TLM and because coincidence of arrival is such an important part of TLM algorithms, this subject has received significantly more attention. Early amongst these was the work of Jaycocks and Pomeroy [6]. However, the first really effective technique which was based on firm theoretical foundations was due to Beyer, Mueller and Hoefer [7] and this will hereafter be referred to as the *BMH* method.

This paper will start with a restatement of the BMH method. We will then present our improved formulation which represents a significant improvement in computational efficiency. Analytical results for the horn antenna [8] will be used as a benchmark against which to compare these boundary-conforming schemes against a conventional TLM model with stepped boundaries.

2. Introduction to the BMH Method

The technique proposed in [7] suggests a recursive definition to describe the arbitrary placed boundaries of the mesh. The definition we are interested in here describes an electric wall (reflection coefficient of $\rho = -1$), by adjusting the incident pulses on the line intersected by the boundary. This is given in formula (5) of the BMH paper [7] as:

$${}_k V^i = \rho \frac{1-\kappa}{1+\kappa} {}_k V^r + \frac{\kappa}{1+\kappa} (\rho_{k-1} V^r + {}_{k-1} V^i) \quad (4)$$

where $\kappa = 2l/\Delta x$, k is the discretised time step and ${}_{k-1} V^r$ ${}_k V^i$ represent the reflected (scattered) and incident pulses at time $k-1$ and k respectively. Figure 3 illustrates the technique graphically. This approach uses a reference plane located at $\Delta x/2$ from a surface adjacent node (broken line in Figure 3). While this is very effective, there are two important shortfalls. As observed from Figure 3, the bounding wall can only cut the transmission line segment after $\Delta x/2$, if the transmission line is intersected before this, it is necessary to remove the node, extending the line from the previous node, causing κ to become greater than 1. While analysis of this has been performed in [7] ensuring the system remains stable, another error is introduced in the system as the connections between neighbouring nodes are now missing. It appears this does not inhibit the accuracy of the technique as much as a stepped approximation would.

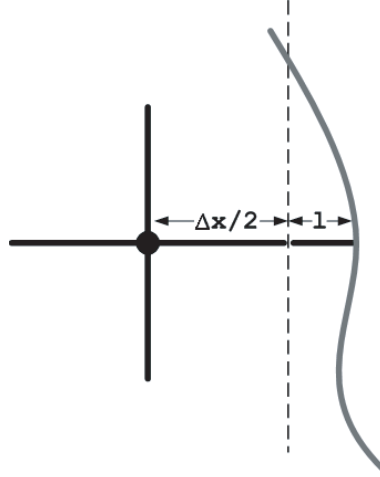
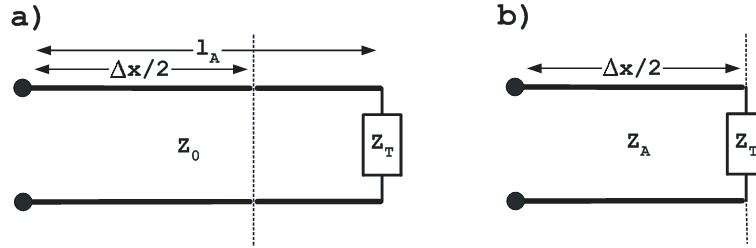


Figure 3: BMH model of arbitrary placed boundary.

Figure 4: a) Apparent impedance of section of length l_A , b) transformed impedance.

The large memory requirements of the recursive procedure proved particularly limiting in the simulations performed in section 4. For a stepped Cartesian mesh using a scatter-collect type algorithm it is necessary to store only 12 scatter and collect matrices (for the 3D case). To implement (4), a further 8 matrices were required to save the previous scattered and incident pulses. While it is possible to only save the data for the boundary nodes, the complexity of the algorithm is further increased.

3. Improved Conforming Boundary Description

The technique we propose avoids the need for recursion, removing the limitations of the BMH approach, while obtaining results in comparison. We begin with a load termination at some arbitrary non-discrete distance from the $\Delta x/2$ line end (Figure 4(a)). Observing the impedance looking ‘down’ this line from the node:

$$Z_{obs} = Z_0 \left[\frac{Z_L + Z_0 \tanh(\beta l_A)}{Z_0 + Z_L \tanh(\beta l_A)} \right] \quad (5)$$

where Z_0 is the intrinsic impedance of the line of length l_A , Z_L represents the load impedance, $\tanh$ is the hyperbolic tangent function, and $\beta = 2\pi/\lambda$, where λ is the wavelength.

As: $Z_0 = \frac{\Delta t/2}{C_d \Delta x/2} = \frac{\Delta t'}{C_d l_A}$, where $\Delta t'$ is the time a signal takes to traverse the line of length l_A we can replace the line of length l_A , impedance Z_0 , with another line of length $\Delta x/2$, with impedance Z_A , as shown in Figure 4(b).

For the case when $\rho = 1$ i.e., $Z_L \rightarrow \infty$ Eq. (5) can be simplified to give, after transformation:

$$Z_{obs} = \frac{Z_A}{\tanh(\beta \Delta x/2)},$$

where $\Delta x/2$ is the discretisation of the model.

Assuming low frequencies $\tanh(\beta \Delta x/2) \approx \beta \Delta x/2$. The impedance transformation observed from the node must be the same before and after transformation:

$$\frac{Z_0}{\beta l_A} = \frac{Z_A}{\beta \Delta x/2}$$

hence:

$$Z_A = Z_0 \left[\frac{\Delta x}{2l_A} \right] \quad (6)$$

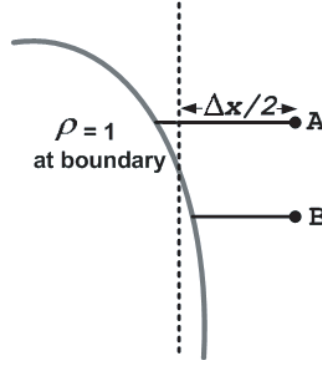


Figure 5: The two cases of impedance transformations covered by (6) and (7).

so if $l_A = \Delta x/2$ $Z_A = Z_0$
 if $l_A > \Delta x/2$ $Z_A < Z_0$ case A in Figure 5
 if $l_A < \Delta x/2$ $Z_A > Z_0$ case B in Figure 5

Likewise for the case when $\rho = -1$ (i.e., $Z_L \rightarrow 0$) (5) simplifies to give, after transformation:

$$Z_{obs} = Z_A \tanh(\beta \Delta x/2), \text{ again assuming } \tanh(\beta \Delta x/2) \approx \beta \Delta x/2,$$

Equating before and after transformation:

$$Z_0 \beta l_A = Z_A \beta \Delta x/2$$

therefore:

$$Z_A = Z_0 \left[\frac{2l_A}{\Delta x} \right] \quad (7)$$

Using both (6) and (7) to describe the boundary adjacent transmission lines for the cases when $\rho = 1$ and $\rho = -1$ respectively, causes the propagating signal to arrive back at the node at time $(k+1)\Delta t$, appearing to have travelled to the true boundary location, while propagating on a line of length $\Delta x/2$.

This scheme will be termed as *uniform* in the analysis performed in section 4.

The nature of mesh-lines at the interface with real surfaces means that we could be dealing with line-lengths in the range $0 < l_A < \Delta x$. Our treatment of this involves expressions with either $\tanh(\beta \Delta x/2)$ or $\tanh \beta \Delta x$. The subsequent analysis assumes that $\tanh \theta \doteq \theta$ so that it is sensible to consider the error bands that are involved. In order to reduce the effects of mesh dispersion conventional TLM in two-dimensions is modelled using discretised frequency $\Delta x/\lambda \leq 0.1$, which means that we are looking at $\lambda \geq 10\Delta x$. If this is the case then

$$\tanh \beta \frac{\Delta x}{2} = \tanh \frac{2\pi}{\lambda} \frac{\Delta x}{2} = \tanh \frac{\pi}{10}$$

The difference between this and $\pi/10$ is 3.16%, a lower bound.

If $l_A \doteq \Delta x$ then our transformation requires that we have $\tanh \beta \Delta x \doteq \beta \Delta x$ so that if we persist with $\Delta x/\lambda \leq 0.1$ then there is an error of 11.37% in this assumption, an upper bound. We can deduce from this that if we operate at $\Delta x/\lambda \leq 0.1/\pi$, then the dispersion at any of the boundary-conforming transmission lines will be no different than if we had used a stepped boundary description with $\Delta x/\lambda \leq 0.1$.

4. Comparison of All Techniques

In order to test the accuracy and viability of this technique, an E-plane sectoral horn antenna has been modelled. The analytical solution to describe the radiated fields from the aperture of the horn is described in Balanis [8]. Figure 6 shows the coordinate system used to describe the dimensions of the horn. The field emitted from the E-plane (y -direction in the TLM models) is given as:

$$E_\theta = -j \left(\frac{a\sqrt{\pi\zeta\rho_1}E_1 e^{-j\zeta r}}{8r} \right) \left\{ -e^{j(\zeta\rho_1 \sin^2 \theta/2)} \left(\frac{2}{\pi} \right)^2 (1 + \cos \theta) F(t_1, t_2) \right\} \quad (8)$$

where ζ denotes the phase factor, E_1 is a constant $F(t_1, t_2) = [C(t_2) - C(t_1)] - j[S(t_2) - S(t_1)]$, $C(t_n)$ and $S(t_n)$ denote the cosine and sine Fresnel integrals:

$$C(t_n) = \int_0^{t_n} \cos\left(\frac{\pi}{2}x^2\right) dx, \quad S(t_n) = \int_0^{t_n} \sin\left(\frac{\pi}{2}x^2\right) dx$$

t_n represents:

$$t_1 = \sqrt{\frac{\zeta}{\pi\rho_1}} \left(-\frac{b_1}{2} - \rho_1 \sin\theta \right), \quad t_2 = \sqrt{\frac{\zeta}{\pi\rho_1}} \left(\frac{b_1}{2} - \rho_1 \sin\theta \right)$$

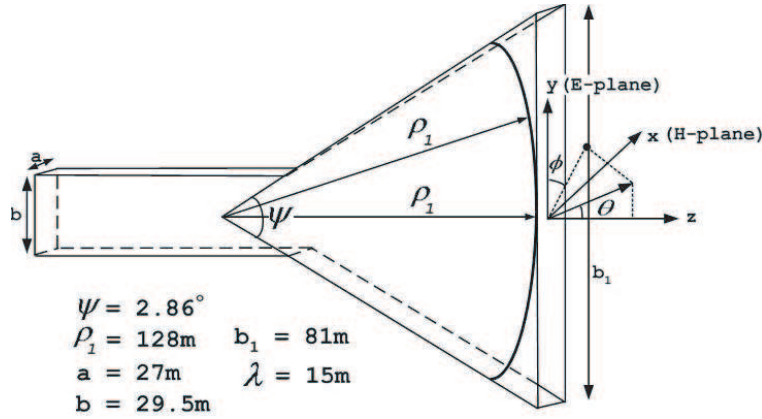


Figure 6: 3D view of E-plane sectoral horn, analytical coordinate system.

Modelling the horn with the dimensions shown in Figure 6, inserting a point source with wavelength of 15 m at the apex of the horn, produces the radiation pattern, along the E-plane, as shown in Figure 7. This has been extracted across the aperture of the horn, from 1/4 into the aperture to 3/4 across (Figure 8), i. e., the centre half of the pattern, this is then plotted over half of the polar diagram. This will act as the benchmark against which to compare the 3D TLM solutions.

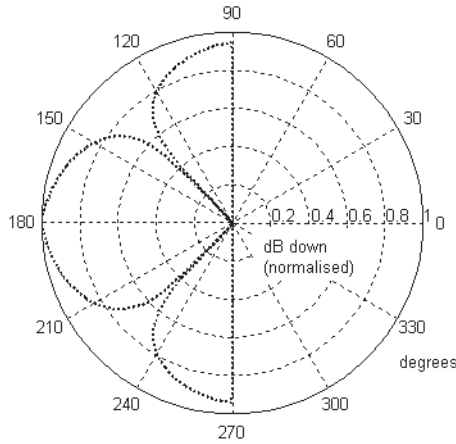


Figure 7: E-plane radiation pattern of E-plane sectoral horn antenna, analytical solution.

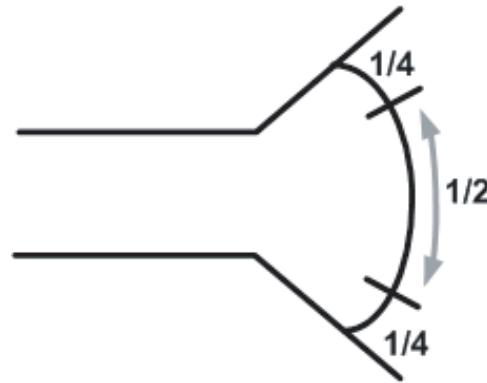


Figure 8: Plotted section of radiated field.

The standard TLM approach creates a 3D Cartesian mesh (for this problem, this is 81 nodes in the north-south direction (y), 128 in the east-west direction (x) and 27 front-back (z)), however as elaborated upon in the discussion given earlier the non-discrete boundaries will become stepped approximations to the true boundary. For this problem the top (south) and bottom (north) boundaries of the mesh will become stepped. The east, west, front and back boundaries are chosen to fall on exact multiples of the models discretisation for the edges of the horn, however to allow the data propagating on the 'corners' of the horn to be included in the simulation, the mouth of the horn has been placed inside the computation space, resulting in east and west boundaries that are also stepped on the flared section of the antennas aperture (Figure 9). Using a reflection coefficient (ρ) of

-1 on all bounding surfaces of the horn and inputting a continuous sinusoidal wave of wavelength 15 nodes (m) at the apex of the horn (marked as $\circ$ in Figure 9), centred in the z direction, the results for the pattern along the aperture of the horn, in comparison to those from the analytical solution are generated in Figure 10. The patterns from the TLM models will never match the analytical solution directly due to the stepwise nature of the computation space. The boundaries are placed at the ends of the transmission lines of length $\Delta x/2$, in comparison to some TLM models which place the boundary at the node. A section of wave-guide of length 150 nodes is appended to the beginning of the model to ensure any errors from the PML have little effect on the signal propagating into the horn. This approach is also used in the BMH and uniform models. A mean error (sum of absolute differences) of 0.0583 is observed, indicating that while the technique produces considerably accurate results given the simplicity of the formulation, they are far from perfect.

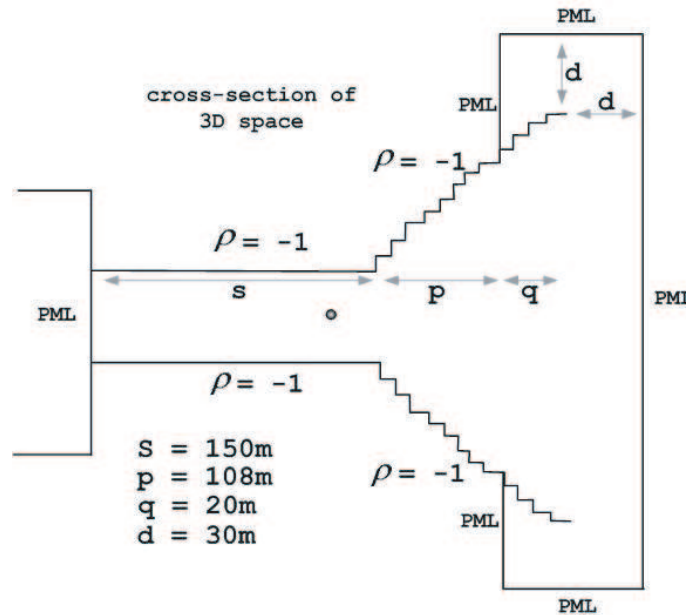


Figure 9: North-south, east-west plane view of 3D E-plane sectoral horn antenna, illustrating TLM stepped formulation.

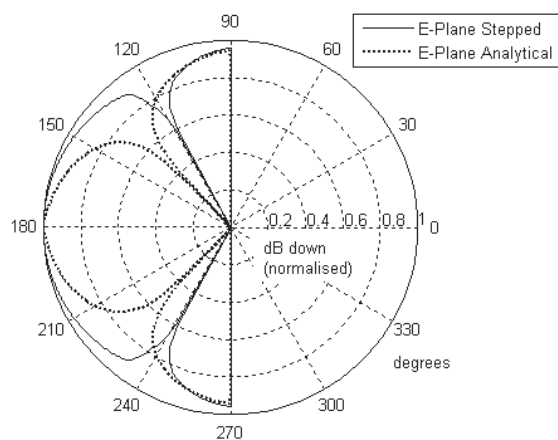


Figure 10: E-plane radiation pattern of E-plane sectoral horn antenna, analytical against stepped TLM (run for 2500 iterations).

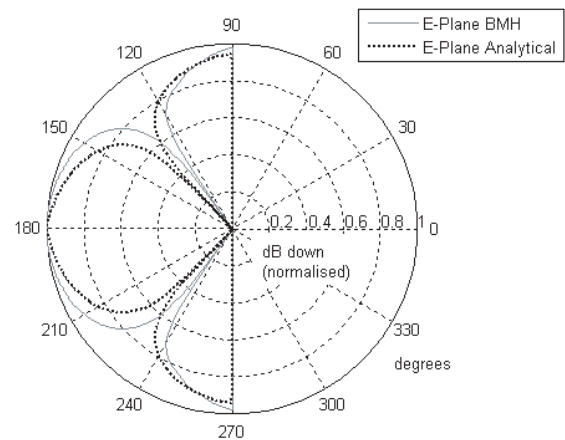


Figure 11: E-plane radiation pattern of E-plane sectoral horn antenna, analytical solution against BMH TLM (run for 788 iterations).

The boundary conforming mesh described in [7] and analysed above was implemented as a comparison to the stepped mesh, again, placing the mouth of the horn inside the computation space, the results which were produced were a considerable improvement on those generated from the stepped mesh and are illustrated in

Figure 11, this is modelled with a reflection coefficient of -1 (or electric wall in the terminology of [7]). As can also be seen the BMH technique requires considerably less iterations than those of the stepped formulation, producing results comparable with the analytical solution after only 788 iterations. The mean error recorded for this mesh was 0.0441.

When the technique that we propose here is implemented, it is clear that its memory requirements are almost identical to those of the stepped mesh. The extra computation needed at the start of the simulation to calculate the lengths of the transmission line segments meeting with the boundaries are usually performed in stepped schemes before the rounding up or down is performed, therefore the only extra computation required in the formulation is the adjustment of the impedances saved in the boundary locations of the impedance matrices. The algorithm then runs in an identical manner to the stepped system. The results produced when this technique was implemented are given in figure 12. The mean error was recorded at 0.0406. Figure 13 shows the uniform scheme in comparison to the BMH results and these are within 0.0035 units of one another, illustrating the accuracy of the new scheme with a substantially smaller computational 'footprint' than the BMH approach.

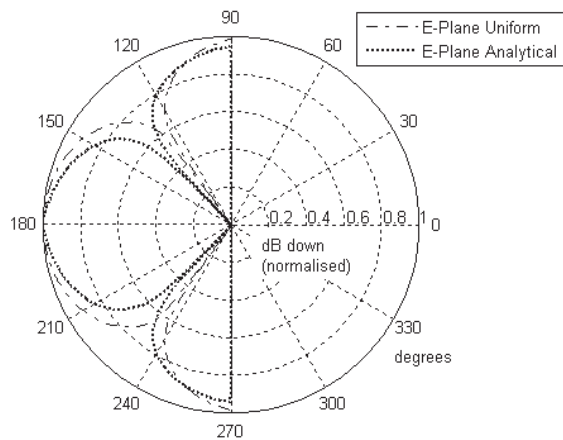


Figure 12: E-plane radiation pattern of E-plane sectoral horn antenna, analytical solution against uniform (run for 2118 iterations).

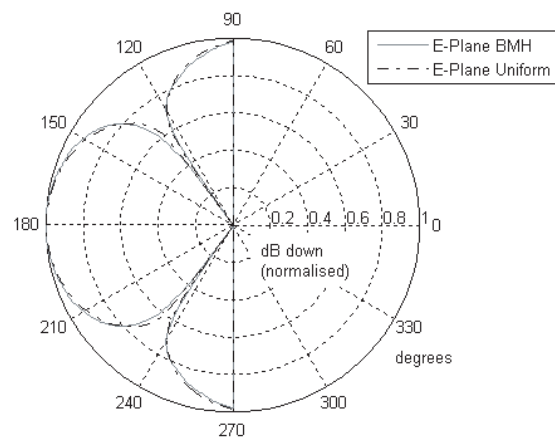


Figure 13: E-plane radiation pattern of E-plane sectoral horn antenna, BMH TLM solution against uniform TLM (BMH run for 788 iterations, uniform run for 2118 iterations).

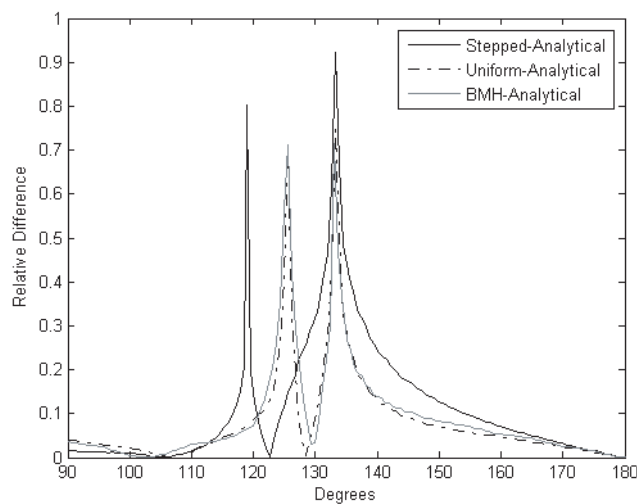


Figure 14: Difference plots of stepped, BMH and uniform TLM models against analytical solution.

Figure 14 gives a graphical view of how close the BMH and uniform models are. Due to the symmetry of the patterns, only half of the plot is shown. As can be observed, the uniform mesh is slightly closer to the analytical solution than the BMH model, while the stepped mesh, as expected, displays significant deviation.

5. Conclusion

The TLM numerical method is widely used, not only in electromagnetics, but many other fields of physics. The technique proposed in this paper gives an accurate approximation to arbitrary placed boundaries of a TLM mesh, while achieving a computational complexity and load equivalent to a normal Cartesian stepped formulation. The method has been compared with another widely used boundary smoothing scheme, illustrating its desirable properties further. The accuracy obtained from the new scheme is in tier with the previously used technique.

We propose this novel approach to model arbitrary placed boundaries of a TLM mesh that do not fall within the discretised formulation of the model. Due to the simplicity of the impedance transformations the computational requirements are practically unaltered from the stepped formulation most commonly used by engineers.

Acknowledgement

The authors would like to acknowledge the assistance and many useful comments which have been provided by Dr. J. Flint, Loughborough University, UK.

REFERENCES

1. Peña, N. and M. M. Ney, "Absorbing-boundary conditions using perfectly-matched-layer (PML) technique for three-dimensional TLM simulations," *IEEE Trans. on Microwave Theory and Techniques*, Vol. MTT-45, 1749–1755, 1997.
2. Tang, T.-W., C. Chrisopoulos, and J. Paul, "Implementation of the stretched co-ordinate based PML for waveguide structures in TLM," *Int. Jnl. of Numerical Modelling*, Vol. 18, 107–118, 2004.
3. De Cogan, D., W. J. O'Connor, and S. Pulko, "Transmission line matrix modelling in computational mechanics," Taylor and Francis/CRC Press, Boca-Raton, 2006.
4. De Cogan, D. and A. de Cogan, "Numerical modelling for engineers", Oxford University Press, 1997.
5. Kreyszig, E., "Advanced engineering mathematics", John Wiley and Sons Inc, Chichester, 1999.
6. Jaycocks, R. and S. Pomeroy, "First international workshop on transmission line matrix (TLM) modelling-theory and applications," University of Victoria, British Columbia (Canada), 269–272, 1–3 August 1995.
7. Mueller, U., A. Beyer, and W. J. R. Hoefer, "Moving boundaries in 2-D and 3-D TLM simulations realized by recursive formulas," *IEEE Trans. on Micro. Theo. and Tech.*, Vol. 40, No. 12, 2267–2271, December 1992.
8. Balanis, C. A., "Antenna theory analysis and design", Harpers and Row Inc, New York, 1982.