EM Field Induced in Inhomogeneous Dielectric Spheres by External Sources

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Abstract—The electromagnetic field induced in the interior of inhomogeneous dielectric bodies by external sources can be evaluated by solving the well-known electric field integrodifferential equation (EFIDE). For spheres with constant magnetic permeability μ, but variable dielectric constant $\varepsilon(r,\theta,\varphi)$ a direct, mainly analytical solution can be used even in case when the inhomogeneity in ε renders separation of variables inapplicable. This approach constitutes a generalization of the hybrid (analytical-numerical) scalar method developed by the authors in two recent papers, for the corresponding acoustic (scalar) field induced in spheres with variable density and/or compressibility. This extension, by no means trivial, owing to the vector and integrodifferential nature of the equation, is based on field-vector expansions using the set of three harmonic surface vectors, orthogonal and complete over the surface of the sphere, for their angular (θ,φ) dependence, and Dini's expansions of a general type for their radial functions. The use of the latter has been shown to be superior to other possible sets of orthogonal expansions and as far as its convergence is concerned it may further be improved by properly choosing a crucial parameter in their eigenvalue equation. The restriction to the spherical shape is imposed here to allow use of the well-known expansion of Green's dyadic in spherical eigenvectors of the vector wave equation.

1. Introduction

The motivation for solving volume integral equations in the case of penetrable (dielectric) spheres with varying dielectric constant $\varepsilon(\vec{r})$ (the magnetic permeability is considered constant throughout) have been discussed thoroughly in a previous paper by the authors [1], dealing with the corresponding scalar problem. The mathematical difficulties of various approaches have been treated in this paper [1], particularly in connection with the advantages of the direct hybrid method proposed here and in [1, 2] by the authors. A first generalization of the approach in acoustics concerned spheres with inhomogeneous density $\rho(\vec{r})$ [2] and herein a further generalization to the vector EM case is developed. In more specific terms we are concerned with the well-known electric field integrodifferential equation (EFIDE)

$$\vec{E}(\vec{r}) = \vec{E}^i(\vec{r}) + \frac{1}{4\pi}(k_0^2 + \nabla\nabla\bullet)\iiint_V \left(\frac{\varepsilon(\vec{r}')}{\varepsilon_0} - 1\right)\vec{E}(\vec{r}')\frac{e^{-ik_0R}}{R}dV' \tag{1}$$

via which the EM field $\vec{E}(\vec{r})$, induced in the interior of an inhomogeneous dielectric body of volume V with varying dielectric constant $\varepsilon(\vec{r})$ (ε_0 is its free space value, while the magnetic permeability μ_0 is considered constant throughout), is evaluated [3–5]. In (1) $\vec{E}^i(\vec{r})$ is the imposed incident field, $R = |\vec{r} - \vec{r}'|$, $k_0 = \omega\sqrt{\mu_0\varepsilon_0} = 2\pi/\lambda$, while $\exp(i\omega t)$ is the assumed time dependence. The induced interior field in V is of primary importance to questions of radiation hazards, to the setting of reliable safety field strength limits in media like living tissue, human heads exposed to nearby EM sources, etc. Following the evaluation of the induced interior field the exterior, scattered one may also be obtained by direct integration.

If V is restricted to be a sphere of radius "a", even when the inhomogeneity $\varepsilon(\vec{r})$ precludes separation of variables, a virtually analytical method can be used to solve (1) based on the possibility of expanding the free space Green's function $G(R) = e^{-ik_0R}/4\pi R$ into an infinite series of spherical eigenfunctions of the Helmholtz equation [1, 2]. This well-known expansion, shown here in the following equation (2) for the corresponding Green's dyadic, is available only in spherical coordinates and combined with Dini-type expansions for the radial functions of the field vectors, provides a basis for a virtually analytical approach. The expansion of Green's dyadic in spherical coordinates is given on page 1875 of [8] in terms of the even/odd spherical eigenvectors of the vector Helmholtz equation. Here we use a more convenient form in terms of the complex form of these vectors as in [9]

$$\bar{\bar{G}}(\vec{r},\vec{r}') = \bar{\bar{I}}\frac{e^{-ik_0|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} = -\frac{ik_0}{4\pi}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\sum_{m=-n}^{n}\frac{(n-m)!}{(n+m)!}[\vec{M}_{mn}^{(1)}(k_0,r_<,\theta,\varphi)\widehat{\vec{M}}_{mn}^{(4)}(k_0,r_>,\theta',\varphi')$$
$$+\vec{N}_{mn}^{(1)}(k_0,r_<,\theta,\varphi)\widehat{\vec{N}}_{mn}^{(4)}(k_0,r_>,\theta',\varphi') + n(n+1)\vec{L}_{mn}^{(1)}(k_0,r_<,\theta,\varphi)\widehat{\vec{L}}_{mn}^{(4)}(k_0,r,\theta',\varphi')] \tag{2}$$

This form can easily be shown to be equivalent to that of [8, 9], where the definitions of the various symbols used here can be found.

2. Solution of the EFIDE

To solve the EFIDE we expand the unknown electric field in vector wave functions in the interval [0, a] in a manner analogous to that of Chew for unbounded media [9, p.397]. The calculation is facilitated by taking into account Gauss's law $\nabla \cdot \vec{D} = 0$, which leads us to write

$$\frac{\varepsilon(\vec{r})}{\varepsilon_0}\vec{E}(\vec{r}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\sum_{l=1}^{\infty}\left[A_{mn\ell}\vec{M}_{mn\ell}\left(\frac{\gamma^M_{mn\ell}}{a},\vec{r}\right) + B_{mn\ell}\vec{N}_{mn\ell}\left(\frac{\gamma^N_{mn\ell}}{a},\vec{r}\right)\right] \tag{3}$$

excluding the vector $\vec{L}$ from the expansion. Similarly we expand the incident field. Here, we have restricted the spectrum of the values of k in the definitions of the vectors $\vec{M}_{mn}(k,\vec{r})$etc, to discrete sets of values $\gamma^M_{mn\ell}$, $\gamma^N_{mn\ell}$, $\ell = 1, 2, ...$, which have been chosen so as to construct a full orthogonal set of vectors $\vec{M}$ and $\vec{N}$, respectively, over the volume of the sphere $0 \le r \le a$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Moreover, all vectors $\vec{M}_{mn\ell}$ and $\vec{N}_{mn\ell}$ in those orthogonal relations are vectors of the first kind, i. e., $\vec{M}_{mn\ell} = \vec{M}^{(1)}_{mn\ell}$, with the upperscript (1) deleted throughout. We can then make use of results like

$$\begin{aligned} I(\vec{M}_{mn\ell},\hat{\vec{M}}_{\mu vp}) &= \int_0^a\int_0^\pi\int_0^{2\pi}\vec{M}_{mn\ell}\left(\frac{\gamma^M_{mn\ell}}{a},\vec{r}\right)\cdot\hat{\vec{M}}_{\mu vp}\left(\frac{\gamma^M_{\mu vp}}{a},\vec{r}\right)r^2\sin\theta drd\theta rd = 4\pi\frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!} \\ &\quad \delta_{m\mu}\delta_{nv}\frac{a^3 j_n(\gamma^M_{mn\ell})j_n(\gamma^M_{mnp})}{(\gamma^M_{mn\ell})^2-(\gamma^M_{mnp})^2}\left[\frac{\gamma^M_{mn\ell}j'_n(\gamma^M_{mn\ell})}{j_n(\gamma^M_{mn\ell})}-\frac{\gamma^M_{mnp}j'_n(\gamma^M_{mnp})}{j_n(\gamma^M_{mnp})}\right], l \ne p \end{aligned} \tag{4}$$

and similar ones for the $\overrightarrow{N}$ and $\overrightarrow{L}$ vectors. Analogous relations for the scalar case were found in [1, 2]. We can now establish full orthogonality of the set over the volume of the sphere by selecting $\gamma^M_{mn\ell}$ as the roots of the "M-eigenvalue equation"

$$\frac{\gamma^M_{mn\ell}j'_n(\gamma^M_{mn\ell})}{j_n(\gamma^M_{mn\ell})} \equiv t^M_{mn}(\ell = 1, 2, \ldots) \tag{5}$$

in which t^M_{nm} may be any chosen constant. Orthogonality of the N-set over the volume of the sphere is, also, established if we choose $\gamma^N_{mn\ell}$ as the roots of the "N-eigenvalue equation"

$$\frac{[\gamma^N_{mn\ell}j_n(\gamma^N_{mn\ell})]'}{(\gamma^N_{mn\ell})^2 j_n(\gamma^N_{mn\ell})} \equiv t^N_{nm}(\ell = 1, 2, \ldots) \tag{6}$$

Finally for the L-set the corresponding $\gamma^L_{mn\ell}$ are chosen as the roots of the "L-eigenvalue equation"

$$\frac{j'_n(\gamma^L_{mn\ell})}{\gamma^L_{mn\ell}j_n(\gamma^L_{mn\ell})} \equiv t^L_{nm}(l = 1, 2, \ldots) \tag{7}$$

Last, but not least, we must establish the orthogonality between the $\vec{L}_{\mu vp}$ and $\vec{N}_{mn\ell}$ sets, which is not assured from their angular part (θ, φ) only. However, over the volume of the sphere we have

$$\begin{aligned} I(\vec{N}_{mn\ell},\hat{\vec{L}}_{\mu vp}) &= \int_V dV'\vec{N}_{mn\ell}(k^N_\ell,\vec{r})\cdot\hat{\vec{L}}_{\mu vp}(k^L_p,\vec{r}) \\ &= 4\pi\frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{m\mu}\delta_{nv}\frac{a^3}{\gamma^N_{mn\ell}\gamma^L_{mnp}}j_n(\gamma^N_{mn\ell})j_n(\gamma^L_{mnp}) \end{aligned} \tag{8}$$

and orthogonality is assured if we choose the roots of $j_n(\gamma^L_{mnp}) = 0$ for the $\vec{L}$ vectors.
We write also

$$\vec{E}(\vec{r}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\sum_{l=1}^{\infty}\left[\Gamma_{mn\ell}\vec{M}_{mn\ell}\left(\frac{\gamma^M_{mn\ell}}{a},\vec{r}\right) + \Delta_{mnl}\vec{N}_{mn\ell}\left(\frac{\gamma^N_{mn\ell}}{a},\vec{r}\right) + Z_{mn\ell}\vec{L}_{mn\ell}\left(\frac{\gamma^L_{mn\ell}}{a},\vec{r}\right)\right] \tag{9}$$

The calculation is carried out with the help of the following intermediate results

$$\begin{aligned} I(\vec{M}\cdot\bar{\bar{I}}g) &= \int_V dV'\vec{M}_{mn\ell}(k,\vec{r}')\cdot\bar{\bar{I}}g(\vec{r},\vec{r}') = \frac{1}{k^2-k_0^2}\{\vec{M}_{mn\ell}(k,\vec{r}) - ik_0a^2[-kj'_n(ka)h_n(k_0a) \\ &\quad +k_0h'_n(k_0a)j_n(ka)]\vec{M}_{mn}(k_0,\vec{r})\} \end{aligned} \tag{10}$$

$$I(\vec{N}\cdot\bar{\bar{I}}g) \;=\; \int\limits_V dV'\vec{N}_{mn\ell}(k,\vec{r}')\cdot\bar{\bar{I}}g(\vec{r},\vec{r}') = \frac{1}{k^2-k_0^2}\{\vec{N}_{mn\ell}(k,\vec{r}) - ik_0a^2[kj_n(ka)\frac{1}{k_0a}\left[xh_n(x)\right]'_{x=k_0a}$$
$$-k_0h_n(k_0a)\frac{1}{ka}\left[xj_n(x)\right]'_{x=ka}]\vec{N}_{mn}(k_0,\vec{r})\} - ik_0a^2n(n+1)\frac{j_n(ka)h_n(k_0a)}{akk_0}\vec{L}_{mn}(k_0,\vec{r}) \tag{11}$$

$$I(\vec{L}\cdot\bar{\bar{I}}g) \;=\; \int\limits_V dV'\vec{L}_{mn\ell}(k,\vec{r}')\cdot\bar{\bar{I}}g(\vec{r},\vec{r}') = \frac{1}{k^2-k_0^2}\{\vec{L}_{mn\ell}(k,\vec{r}) - ik_0a^2[kj_n(ka)h_n'(k_0a)$$
$$-k_0h_n(k_0a)j_n'(ka)]\vec{L}_{mn}(k_0,\vec{r})\} - ik_0a^2\frac{j_n(ka)h_n(k_0a)}{akk_0}\vec{N}_{mn}(k_0,\vec{r}) \tag{12}$$

In all the above equations $k = \frac{\gamma^X_{mnl}}{a}$, with $X = M$ or N or L respectively.

After lengthy manipulations we get the system of equations

$$\Gamma_{mn\ell} \;=\; A^{inc}_{mn}T^M_{n\ell}(k_0) + k_0^2\frac{A_{mn\ell}-\Gamma_{mn\ell}}{\left(\frac{\gamma^M_{mn\ell}}{a}\right)^2 - k_0^2}$$
$$-ik_0^3a^2T^M_{n\ell}(k_0)\sum_{\mathrm{p}}\frac{\left[-\frac{\gamma^M_{mnp}}{a}j_n'(\gamma^M_{mnp})h_n(k_0a) + k_0h_n'(k_0a)j_n(\gamma^M_{mnp})\right]}{\left(\frac{\gamma^M_{mnp}}{a}\right)^2 - k_0^2}(A_{mnp}-\Gamma_{mnp}) \tag{13}$$

$$\Delta_{mn\ell} \;=\; B^{inc}_{mn}T^N_{n\ell}(k_0) + k_0^2\frac{B_{mn\ell}-\Delta_{mn\ell}}{\left(\frac{\gamma^N_{mn\ell}}{a}\right)^2 - k_0^2} - ik_0^3a^2T^N_{n\ell}(k_0)\cdot$$
$$\sum_p\frac{\left[\frac{\gamma^N_{mnp}}{a}j_n(\gamma^N_{mnp})\frac{1}{k_0a}\left[xh_n(x)\right]'_{x=k_0a} - k_0h_n(k_0a)\frac{1}{\gamma^N_{mnp}}\left[xj_n(x)\right]'_{x=ka}\right]}{\left(\frac{\gamma^N_{mnp}}{a}\right)^2 - k_0^2}(B_{mnp}-\Delta_{mnp}) \tag{14}$$

where $T^M_{n\ell}(k_0)$ and $T^N_{n\ell}(k_0)$ are obviously the expansions coefficients of $\vec{M}_{mn}(k_0,\vec{r})$, $\vec{N}_{mn}(k_0,\vec{r})$ over the orthogonal sets $\vec{M}_{mn\ell}$ and $\vec{N}_{mn\ell}$, $\ell = 1, 2, ...$, respectively.

The next step is to eliminate one of the two groups of unknowns {A, B} or $\{\Gamma, \Delta, \mathrm{Z}\}$. Since Z are not present in the final expressions it is better to eliminate Γ, Δ, so we write

$$\vec{E} = \frac{1}{\frac{\varepsilon(\vec{r})}{\varepsilon_0}}\sum_{m,n,\ell}\left[A_{mn\ell}\vec{M}_{mn\ell} + B_{mn\ell}\vec{N}_{mn\ell}\right] = \sum_{m,n,\ell}\left[\Gamma_{mn\ell}\vec{M}_{mn\ell} + \Delta_{mn\ell}\vec{N}_{mn\ell} + Z_{mn\ell}\vec{L}_{mn\ell}\right] \tag{15}$$

We have thus constructed the necessary equations which, via the orthogonality relations, lead to a matrix equation for the unknowns A, B.

3. Numberical Results and Discussion

Numerical results were obtained for both radial and r-,θ- dependence of the inhomogeneities. For the simpler r-case we selected the well known case of Eaton lens, $\varepsilon(r) = (r/a)^2$ [11, 12]. We have then reproduced exactly the same results known from the literature.

We next present results for the more complicated $r-$, $\theta-$ case. Here we have worked without optimizing the t_{mn} values (a complicated problem) and without comparison to existing results, that are lacking in this more general case. However, a confirmation of the correctness of our procedure stems from the reproduction of our results with other random choices for t_{mn}.

For convenience, that is to obtain as many intermediate results as possible in analytic form and reduce the numerical burden, we have chosen the following function

$$\varepsilon(\vec{r}) = \frac{\varepsilon_0}{1 + 0.3\left(\frac{r}{a}\right)^2\cos\theta} \tag{16}$$

Here, working with $k_0a = 2.0958$ and using as incident field a plane wave $\hat{x}e^{ikz}$ [10], we present final results for the total interior field $|\vec{E}^{tot}(r,\,\theta,\,\varphi=0)/E_0|$($E_0$ is the amplitude of $\vec{E}_{inc}$) for a few particular values of θ, which is treated as a parameter in the Figure. Our results correspond to $\varphi = 0$. It turned out that we should take $n = 5$ terms for the M-component and $n = 10$ terms for the N-component. In all cases we used $\ell_T = 12$ terms and this proved to be sufficient.

The maximum value of the total interior field appears at $r/a = 0.95$ and $\theta = 77°$.

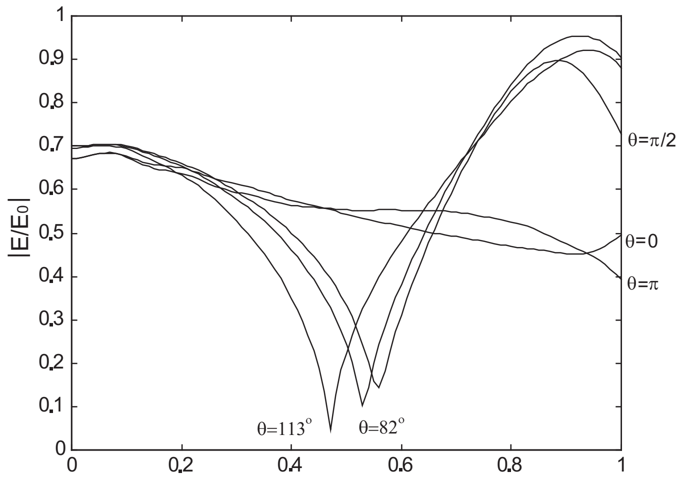

Figure 1: $|E|$ for $\varepsilon(\vec{r}) = \varepsilon_0 \left(1 + 0.3 \left(\frac{r}{a}\right)^2 \cos\theta\right)^{-1}$, $k_0 a = 2.0958$ for various θ, incident field $\hat{x}e^{ikz}$.

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