

# Arbitrary Lagrangian Eulerian Electromechanics in 3D

R. Rieben, B. Wallin, and D. White  
Lawrence Livermore National Laboratory, USA

**Abstract**—We present results from an effort to couple the equations of electromagnetic diffusion with the equations of arbitrary Lagrangian-Eulerian (ALE) hydrodynamics. The electromagnetic diffusion equations are discretized using a novel mixed finite element method coupled with a generalized Crank-Nicholson time differencing scheme. At each discrete time step, electromagnetic force and heat terms are calculated and coupled to the hydrodynamic equations in an operator split approach. We present preliminary results from a fully coupled electromechanical simulation as well as results concerning advection techniques for electromagnetic quantities.

## 1. Introduction

We are interested in the simulation of electromechanical devices and magnetohydrodynamic events in three dimensions. Our primary goal is a numerical method that solves, in a self-consistent manner, the equations of electromagnetics (primarily statics and diffusion), heat transfer (primarily conduction), and non-linear mechanics (elastic-plastic deformation, and contact with friction). In this paper, we focus on the numerical discretization of electromagnetic diffusion in an arbitrary Lagrangian-Eulerian (ALE) fashion for the purposes of computing  $\vec{J} \times \vec{B}$  forces for mechanical (or hydrodynamic) calculations and  $\vec{J} \cdot \vec{E}$  Joule heating terms for thermal calculations.

The equations of electromagnetic diffusion can be derived from the full wave Maxwell equations by making the good conductor approximation (i. e., ignoring displacement current), which is standard practice in magnetohydrodynamic (MHD) formulations. For conducting materials moving with a velocity  $\vec{v}$  with respect to a fixed Eulerian (or laboratory) frame, we can derive the so called dynamo equation (also known as the hydromagnetic equation) in terms of magnetic flux density

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \left( \frac{1}{\sigma} \vec{\nabla} \times \frac{1}{\mu} \vec{B} \right) + \vec{\nabla} \times (\vec{v} \times \vec{B}) \quad (1)$$

In the Eulerian description the velocity  $\vec{v}$  is a function of time  $t$  and position  $\vec{x}$ . In the *Lagrangian* (or material) description (which we will designate with a “prime” symbol), the flow is described by following the position  $\vec{x}'(\vec{x}', t)$  of the material point that started at position  $\vec{x}'$  at  $t = 0$ . In functional form, we have

$$\vec{x}' = \vec{x}'(\vec{x}, t); \quad \vec{x} = \vec{x}(\vec{x}', t)$$

To convert between the two representations, we define the Jacobian matrix as

$$J_{i,j} = \frac{\partial x'_j}{\partial x_i} \quad (2)$$

As shown in [1], the following quantities are invariant with respect to the Lagrangian-Eulerian representations

$$\begin{array}{ll} \text{Lagrangian} & \text{Eulerian} \\ \vec{E}' \cdot d\vec{x}' & = (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{x} \\ \vec{B}' \cdot d\vec{\alpha}' & = \vec{B} \cdot d\vec{\alpha} \end{array} \quad (3)$$

It is well known that differential arc length and surface area elements transform according to

$$d\vec{x}' = J^T d\vec{x} \quad (4)$$

$$d\vec{\alpha}' = |J| J^{-1} d\vec{\alpha} \quad (5)$$

As a consequence, the electric field intensities and magnetic flux densities must transform inversely to maintain the invariance property of (3)

$$\vec{E}' = J^{-1}(\vec{E} + \vec{v} \times \vec{B}) \quad (6)$$

$$\vec{B}' = \frac{1}{|J|} J^T \vec{B} \quad (7)$$

The dynamo equation in the Lagrangian frame is therefore

$$\frac{d\vec{B}'}{dt} = -\vec{\nabla}' \times \left( \frac{1}{\sigma} \vec{\nabla}' \times \frac{1}{\mu} \vec{B}' \right) \quad (8)$$

In a typical ALE hydrodynamic calculation, an *operator split* method is employed where all calculations are performed on a Lagrangian mesh (i. e., a mesh that moves with the materials). When the Lagrange motion of the mesh causes significant mesh distortion, that distortion is corrected with an equipotential relaxation of the mesh, followed by a 2nd order monotonic remap of mesh quantities. This remap is equivalent to an advection of material through the mesh. In our proposed ALE formulation of MHD, we will employ an operator-split method with three distinct steps:

- *Electromagnetic Diffusion*—Solve the dynamo equation in the Lagrangian frame at one discrete time step for fixed materials.
- *Lagrangian Motion*—Move mesh nodes according to  $\vec{J}' \times \vec{B}'$  forces assuming a  $\frac{d\vec{B}'}{dt} = 0$  “frozen flux” condition.
- *Eulerian Advection*—Only required if mesh is relaxed, advect (or transport) magnetic (vector potential) flux quantities to new mesh.

Note that the second step (effectively “dragging” the electromagnetic quantities along with the mesh during Lagrangian motion) will only work if our discretization of the electromagnetic quantities satisfies the invariance relation of (3) (see also [2]). In the Eulerian advection step of the calculation, the computed electromagnetic degrees of freedom must be “remapped” or “advected” in a way which preserves a discrete divergence-free property of the magnetic flux density with minimal magnetic energy loss.

## 2. Numerical Formulation

The divergence-free (or solenoidal) nature of the magnetic flux density,  $\vec{\nabla}' \cdot \vec{B}' = 0$ , implies that  $\vec{B}' = \vec{\nabla}' \times \vec{A}'$  where  $\vec{A}'$  is a magnetic vector potential. This in turn implies that the electric field in the Lagrangian frame is given by  $\vec{E}' = -\vec{\nabla}'\phi' - \frac{\partial}{\partial t}\vec{A}'$ , where  $\phi'$  is an electric scalar potential. Using the gauge condition  $\vec{\nabla}' \cdot \sigma\vec{A}' = 0$ , we can reformulate the dynamo equation (8) in terms of potentials as

$$\vec{\nabla}' \cdot \sigma\vec{\nabla}'\phi' = 0 \quad (9)$$

$$\sigma \frac{d\vec{A}'}{dt} = -\vec{\nabla}' \times \frac{1}{\mu} \vec{\nabla}' \times \vec{A}' - \sigma\vec{\nabla}'\phi' \quad (10)$$

Note that this formulation has an additional elliptic PDE (9) to solve for the scalar potential. A key advantage of this formulation is that voltage, which is often the only known quantity for electromechanical engineering applications, appears explicitly in the equations as an essential boundary condition for the elliptic solution of (9). To compute force and heat terms, we define the *secondary variables* in terms of the potentials as

$$\vec{B}' = \vec{\nabla}' \times \vec{A}' \quad (11)$$

$$\vec{J}' = \sigma\vec{E}' = -\vec{\nabla}'\phi' - \frac{d}{dt}\vec{A}' \quad (12)$$

Finally, there are divergence constraints on both the primary and secondary fields, namely

$$\vec{\nabla}' \cdot \sigma\vec{A}' = 0 \quad (13)$$

$$\vec{\nabla}' \cdot \vec{B}' = 0 \quad (14)$$

To discretize the potential formulation in the Lagrangian frame, we apply the mixed finite element methods (FEM) of [3] which are based on the properties of differential forms and have been shown to preserve discrete divergence-free properties and to maintain accuracy in secondary variables (e.g.,  $\vec{J}$  and  $\vec{B}$ ) even when computed from potentials. Most importantly, the discrete vector fields transform identically to (6) and (7), thereby preserving the invariance property of (3).

In our proposed ALE formulation the scalar potential will be discretized on mesh nodes (i.e., a discrete 0-form field), the vector potential will be discretized on mesh edges (i.e., a discrete 1-form field) and the secondary variables  $\vec{B}$  and  $\vec{J}$  will be discretized on mesh faces (i.e., discrete 2-form fields) as follows

$$\phi' \approx \sum_{i=1}^n v_i W_i^0 \quad (15)$$

$$\vec{A}' \approx \sum_{i=1}^n a_i \vec{W}_i^1 \quad (16)$$

$$\vec{B}' \approx \sum_{i=1}^n b_i \vec{W}_i^2 \quad (17)$$

$$\vec{J}' \approx \sum_{i=1}^n j_i \vec{W}_i^2 \quad (18)$$

where  $W^l$  denotes a discrete  $l$ -form basis function. In [3], various mass, stiffness, derivative and discrete Hodge matrices are defined. Given these matrices, the fully discrete form of the potential diffusion equation is given in [3] by applying a Generalized Crank-Nicholson method to obtain

$$\mathbf{S}^0 \mathbf{v}_{n+\alpha} = \mathbf{f}_{n+\alpha}^0 \quad (19)$$

$$(\mathbf{M}^1(\sigma) + \alpha \Delta t \mathbf{S}^1(\mu^{-1})) \mathbf{a}_{n+1} = (\mathbf{M}^1(\sigma) - (1 - \alpha) \Delta t \mathbf{S}^1(\mu^{-1})) \mathbf{a}_n - \Delta t \mathbf{D}^{01} \mathbf{v}_{n+\alpha} \quad (20)$$

where  $\alpha \in [0,1]$  is weighting parameter which determines the type of integration such that

$$\alpha = \begin{cases} 0 & \text{Explicit, 1st Order Accurate Forward Euler} \\ 1/2 & \text{Implicit, 2nd Order Accurate Crank Nicholson} \\ 1 & \text{Implicit, 1st Order Accurate Backward Euler} \end{cases}$$

Once the values for the primary potentials have been solved for, the discrete secondary fields can be computed as

$$\mathbf{e}_{n+\alpha} = -\mathbf{K}^{01} \mathbf{v}_{n+\alpha} - 1/\Delta t (\mathbf{a}_{n+1} - \mathbf{a}_n) \quad (21)$$

$$\mathbf{b}_{n+1} = \mathbf{K}^{12} \mathbf{a}_{n+1} \quad (22)$$

$$\mathbf{M}^2(\sigma^{-1}) \mathbf{j}_{n+\alpha} = \mathbf{H}^{12} \mathbf{e}_{n+\alpha} \quad (23)$$

These terms are used to compute  $\vec{J}' \times \vec{B}'$  forces which will accelerate the mesh nodes during the Lagrangian motion step. The discrete divergence constraints are given by

$$(\mathbf{D}^{01}(\sigma))^T \mathbf{a} = 0 \quad (24)$$

$$(\mathbf{D}^{01}(\sigma))^T \mathbf{e} = 0 \quad (25)$$

$$\mathbf{K}^{23} \mathbf{b} = 0 \quad (26)$$

and as shown in [3], these constraints are implicitly satisfied for all time, assuming the initial conditions and the source terms are divergence free.

To demonstrate a fully coupled Lagrangian calculation, we consider a numerical experiment in which a 5 KV capacitor bank is discharged into a can shaped aluminum structure (see Fig. 1). The voltage through the can (effectively an inductive and resistive load) is computed via a simple SPICE model. The resulting voltage vs. time profile is then used as an essential boundary condition for the discrete scalar potential solve of (19) which

drives the problem. An essential boundary condition of the form  $\hat{n} \times \vec{A}' = 0$  is applied to the front side of the mesh while the remainder of the surface is subject to the natural boundary condition  $\hat{n} \times \frac{1}{\mu} \vec{\nabla}' \times \vec{A}' = 0$ . A peak current of roughly 0.8 MA is generated in the can, creating a  $\vec{J}' \times \vec{B}'$  force which causes the can to initially compresses (or implode). However, the force is not strong enough to cause the aluminum can to yield, and so the can effectively rings over time in an elastic response as shown in Figs. 1 and 2.

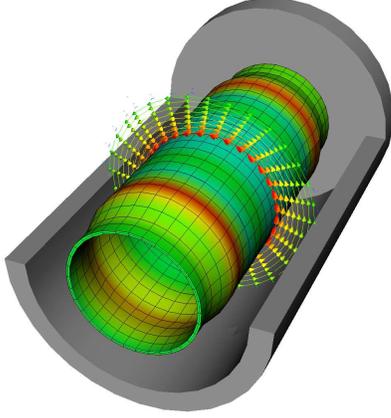


Figure 1: Snapshot of the fully coupled electromechanical simulation. In this image the aluminum can is elastically expanding after initially being compressed. The displacement has been exaggerated by a factor of 300 for visual clarity.

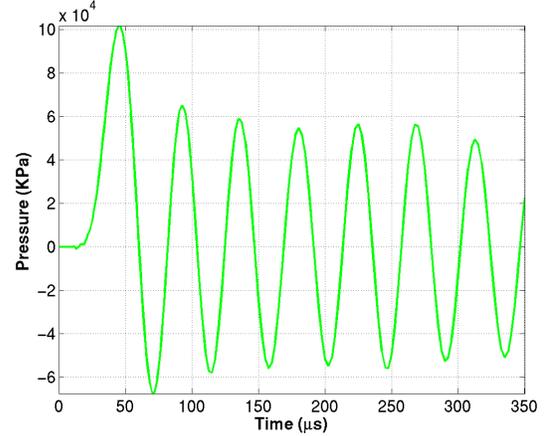


Figure 2: Measured pressure response in the aluminum can due to an electromagnetic force.

### 3. Constrained Transport Methods on Unstructured Grids

During the optional Eulerian advection phase of our operator split method, the computed electromagnetic values must be remapped (or advected). Remapping refers to the process of updating the representation of the field given a new grid. We consider only new grids which are “nearby” in the sense that only small perturbations of the grid are allowed (i. e., the mesh nodes should not travel farther than one mesh element in any one time step). This is known as the continuous remap approximation (CRA).

We propose to use the so called *constrained transport* method originally developed by [4] and later expanded by [5]. Suppose we have calculated the magnetic flux density  $\vec{B}'$  in a Lagrangian time step via (22), we then have a local element representation of  $\vec{B}'$

$$\vec{B}^{old} \approx \sum_{i=1}^n b_i^{old} \vec{W}_i^{2,old} \quad (27)$$

The degrees of freedom (DOF)  $b_i^{old}$  in this expansion carry the units of magnetic *flux*. For the special case of lowest order ( $p = 1$ ) basis functions (i. e., six DOF per element), this implies that we know the magnetic flux through every face in the Lagrangian mesh (or the “old” mesh). Now in a standard ALE step, the old mesh is relaxed under the CRA to a *new* mesh. Therefore, our goal is to compute *new* values of the magnetic flux  $b_i^{new}$  which will allow us to represent the magnetic flux density on the new mesh. For the special case of lowest order ( $p = 1$ ) basis functions, the discrete divergence free property is simply a statement that the 6 fluxes in the face sum to zero. The goal of *constrained* transport is to preserve this property on the new mesh.

For unstructured hexahedral grids, we can update the magnetic flux (or “vector potential flux”  $\vec{A} \cdot d\vec{x}$ ) by effectively solving Faraday’s law for a moving conductor (equivalent to magnetic transport under the “frozen-flux” condition)

$$\Phi^{new} \approx \Phi^{old} - \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l} \quad (28)$$

where  $\vec{u}$  is the mesh displacement. Our goal now is to apply (28) in an algorithmic fashion to update the fluxes on the faces of a new mesh. A schematic representation of this process is shown in Fig. 3. It is clear from

the depiction of Fig. 3 that we can approximate the flux through a new face given the flux through the old face and a “measurement” of the time rate of change of flux (an effective voltage) along the closed circuit path  $C$  depicted in green. For a given face in the new mesh, the algorithm of (28) can be used to update the edge flux contributions

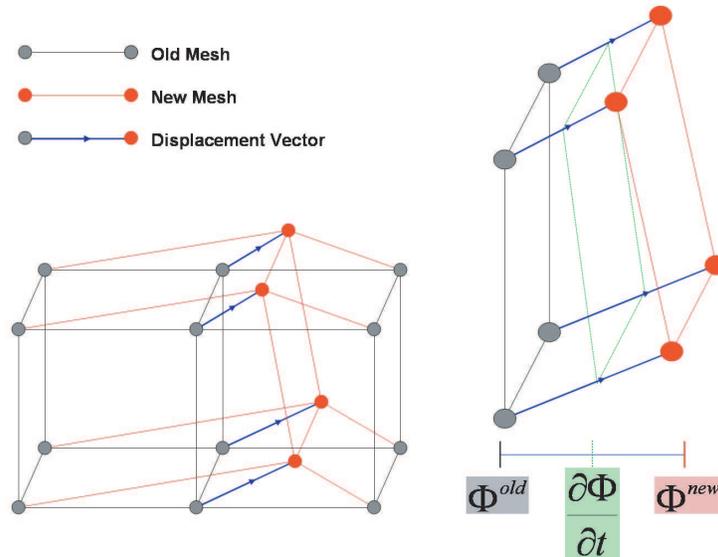


Figure 3: Schematic diagram depicting the relationship between magnetic flux through “old” and “new” mesh faces and the most accurate location for “measuring” the update EMF.

$\vec{A} \cdot d\vec{x}$  for each edge in the face (thereby updating the vector potential) or the total magnetic flux  $\vec{B} \cdot d\vec{a}$  through the face. By construction, the new flux values will sum to zero, provided the old fluxes do so as well. In order for this algorithm to work,  $\vec{B}$  must be evaluated at the displacement vector midpoints for the discrete path integral; however, this is problematic for faced based representations of  $\vec{B}$ , since they are by construction, discontinuous along element edges. To overcome this, a “smooth”  $\vec{B}$  field must be patch recovered using a continuous vector nodal approximation.

#### 4. Conclusions

We have presented and discussed an operator split approach for solving the coupled equations of electromechanics and magnetohydrodynamics using the novel mixed finite element methods of [3] to discretize the equations of electromagnetic diffusion. We have presented preliminary results for a fully coupled Lagrangian calculation and have discussed methods for advecting magnetic flux for ALE calculations.

#### Acknowledge

This work was performed under the auspices of the U.S. Department of Energy by the University of California, Lawrence Livermore National Laboratory under contract No. W-7405-Eng-48, UCRL-ABS-215152.

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