Statistical Distribution of Field Scattered by 1-Dimensional Random Slightly Rough Surfaces

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Abstract—We consider a perfectly conducting plane with a local cylindrical perturbation illuminated by a monochromatic plane wave. The perturbation is represented by a random function assuming values with a Gaussian probability density. For each realization of the stochastic process, the spatial average value over the width of the modulated zone is zero. The mean value of the random function is also zero. Without any deformation, the total field is the sum of the incident field and the reflected field. For a locally deformed plane, we consider — in addition to the incident and reflected plane waves — a scattered field. Outside the modulated zone, the scattered field can be represented by a superposition of a continuous spectrum of outgoing plane waves. The method of stationary phase leads to the asymptotic field, the dependence angular of which is given by the scattering amplitudes of the propagating plane waves. Using the first-order small perturbation method, we show that the real part and the imaginary part of scattering amplitudes are uncorrelated Gaussian stochastic variables with zero mean values and unequal variances. Consequently, the probability density for the amplitude is given by the Hoyt distribution and the phase is not uniformly distributed between 0 and 2π.

1. Introduction

The problem of electromagnetic wave scattering from random surfaces continues to attract research interest because of its broad applications. The three classical analytical methods commonly used in random rough-surface scattering are the small-perturbation method, the Kirchhoff method and the small slope approximation [1–5]. The electromagnetic analysis of rough surfaces with parameters close to the incident wavelength requires a rigorous formalism. Numerous method based on Monte Carlo simulations are available for 1D and 2D random rough surfaces [6, 7]. Most of research works focus on the determination of coherent and incoherent intensities. There is not such a voluminous literature on the statistical distribution of scattered field [3]. In this paper, we derive the statistical distribution in the far field zone from the first-order small perturbation method in the particular case of perfectly conducting 1D random rough surface illuminated by an $E_{//}$ polarized monochromatic plane wave.

2. The Random Surfaces under Consideration

The geometry of the problem is depicted in Fig. 1. The rough surface is represented in Cartesian coordinates by the equation $y = a_0(x)$ and consists of a small cylindrical random perturbation with length L and zero mean

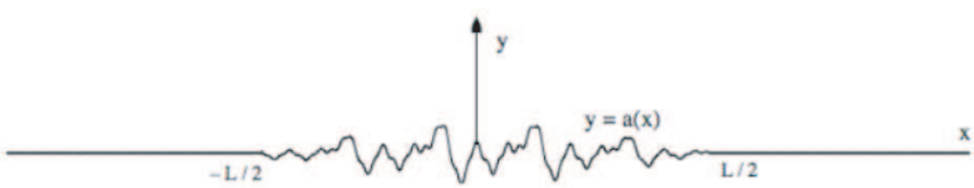

Figure 1: The slightly rough surface.

($< a_0(x) >= 0$) in a perfectly conducting plane $y = 0$. Each realisation can be described by the following equation

$$\begin{aligned} a_0(x) &= a(x) - m \qquad \text{if } |x| \le \frac{L}{2} \\ a_0(x) &= 0 \text{ outside} \end{aligned} \tag{1}$$

where

$$m = \frac{1}{L} \int_{-L/2}^{+L/2} a(x) dx \tag{2}$$

$a(x)$ is a random function assuming values distributed normally with zero mean and variance σ_a^2. Here it's important to distinguish the spatial average m from the statistical mean $< a(x) >$. Insofar $< a(x) >= 0$, we have $< m >= 0$. The random process is assumed stationary with a Gaussian statistical correlation function

$$R_{aa}(x) = \sigma_a^2 \exp\left(- \frac{x^2}{l_c^2} \right) \tag{3}$$

where l_c is the correlation length.

3. The Scattering Amplitudes in the Far Field Zone

The surface is illuminated under incidence θ_i by an z-polarized monochromatic plane wave $E_i \overrightarrow{u}_z$ of wavelength λ. The Oz-electric component of field is

$$E_i(x, y) = \exp(-j\alpha_i x + j\beta_i y) \tag{4}$$

where

$$\alpha_i = k \sin\theta_i\,; \quad \beta_i = k\cos\theta_i\,; \quad k = 2\pi/\lambda \tag{5}$$

The time-dependence factor $\exp(j\omega t)$ where ω is the angular frequency is assumed and suppressed throughout. The total electric field above the rough surface is the sum of the incident field E_i, the field reflected E_r by the plane without deformation (an infinite perfect mirror) and the scattered field E_d.

$$E_t(x, y) = E_i(x, y) + E_r(x, y) + E_d(x, y) \tag{6}$$

where

$$E_r(x, y) = -\exp(-j\alpha_i x - j\beta_i y) \tag{7}$$

Above the highest point on the surface, the scattered field can be represented by a superposition of a continuous spectrum of outgoing plane waves, the so-called Rayleigh integral [5].

$$E_d(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{C}(\alpha) \exp\big(-j\beta(\alpha)y\big) \exp(-j\alpha x) d\alpha \tag{8}$$

with

$$\beta = \sqrt{k^2 - \alpha^2}, \quad \operatorname{Im}\beta < 0 \tag{9}$$

In the far-field zone, the Rayleigh integral is reduced to the only contribution of the propagating waves ($\alpha \leq k$). The method of stationary phase leads to the asymptotic field [8]

$$E_d(r, \theta) \approx \sqrt{\frac{k}{2\pi r}} \hat{C}(k\sin\theta)\cos\theta \exp(-jkr) \exp\left(j\frac{\pi}{4}\right) \tag{10}$$

The angular dependence in the far field zone is given by the function $\hat{C}(\alpha)\cos\theta$ and becomes identified with the propagating wave amplitudes of the continuous spectrum (8) with $\alpha = k\sin\theta$ [9, 10]. Let us recall that the normalized bistatic scattering coefficient $\sigma(\theta)$ is defined by the power scattered per unit angle $d\theta$ normalized with respect to the flux of incident power through the modulated region

$$\sigma(\theta) = \frac{1}{P_i}\frac{dP_d}{d\theta} = \frac{|\hat{C}(k\sin\theta)|^2\cos^2\theta}{\lambda L\cos\theta_i} \tag{11}$$

For a random process, the scattered field is a random function of position (r, θ) but the scattering amplitude $\hat{C}(\alpha)$ is a random function of the observation angle θ only [10]. The scattering amplitude can be written as the sum of an average amplitude $< \hat{C}(\alpha) >$ which gives the coherent far-field from (11) and a fluctuating amplitude which leads to the incoherent far-field. The first order small perturbation method applied to the Rayleigh integral (8) and the Dirichlet boundary condition gives an approximation of the scattering amplitudes [1, 2]

$$\hat{C}(\alpha) = -2j\beta_i \int_{-L/2}^{+L/2} a_0(x) \exp\big(+j(\alpha - \alpha_i)x\big) dx \tag{12}$$

Making a change of variable $\gamma = \alpha - \alpha_i$, real part $\hat{C}_r(\alpha)$ and imaginary part $\hat{C}_i(\alpha)$ of scattering amplitudes can be expressed as

$$\hat{C}_r(\gamma) \; = \; +2\beta_i \int_{-L/2}^{+L/2} a(x)\sin(\gamma\, x)dx \tag{13}$$

$$\hat{C}_i(\gamma) \; = \; -2\beta_i \Bigg(\int_{-L/2}^{+L/2} a(x)\cos(\gamma\, x)dx - mL\sin c(\gamma\, L/2) \Bigg) \tag{14}$$

where $\sin c(x) = \sin x/x$. It can be noticed that the scattering amplitude is zero in the specular direction $\gamma = 0$. $\hat{C}_r$ and $\hat{C}_i$ are obtained from mathematical linear operations applied to the Gaussian random function $a(x)$. Consequently, $\hat{C}_r$ and $\hat{C}_i$ are also quantities distributed with Gaussian probability densities.

4. The Statistical Distribution of Scattering Amplitudes

4.1. The Incoherent Intensity

From (13) and (14), we derive $< \hat{C}(\gamma) >= 0$. Consequently, the coherent density is zero. Moreover, after some extensive mathematical manipulations, we deduce the variances

$$r =< \hat{C}_r^2 > \; = \; 4\beta_i^2 \int_0^{+L} (L-x)\Big[\cos\gamma x - \sin c\big(\gamma(L-x)\big)\Big] R_{aa}(x)dx \tag{15}$$

$$\begin{aligned} s =< \hat{C}_i^2 > \; &= \; 4\beta_i^2 \int_0^{+L} (L-x)\Big[\cos\gamma x + \sin c\big(\gamma(L-x)\big)\Big] R_{aa}(x)dx - 4\beta_i^2 \sin c(\gamma\, L/2) \\ &\Bigg[\sin c(\gamma\, L/2) \int_0^{+L} xR_{aa}(x)dx + 2\int_0^{+L} (L/2-x)\sin c\big(\gamma(L/2-x)\big) R_{aa}(x)dx\Bigg] \end{aligned} \tag{16}$$

where the statistical correlation function $R_{aa}(x)$ is given by (3).

The variances depend on the width L of the modulated zone. But, outside the specular reflection zone, if L goes to infinity, the variances of the real and imaginary parts become identified. Using (11), (15) and (16), we obtain the incoherent intensity $I_f(\theta) =< \sigma(\theta) >$

$$I_f(\theta) = \frac{< \big|\hat{C}(k\sin\theta - k\sin\theta_i)\big|^2 > \cos^2\theta}{\lambda L\cos\theta_i} \quad \text{with} < \big|\hat{C}(\gamma)\big|^2 >=< \hat{C}_r^2 > + < \hat{C}_i^2 > \tag{17}$$

We note that the incoherent intensity is not proportional to the surface power spectrum.

4.2. Probability Densities of the Amplitude and Phase

Random quantities $A = \hat{C}_r(\alpha)$ and $B = \hat{C}_i(\alpha)$ are distributed normally with zero mean values and unequal variances r and s. Moreover, we show that they are uncorrelated. Consequently, they are independent and we can write:

$$p_{AB}(a,b) = p_A(a)p_B(b) = \frac{1}{2\pi\sqrt{rs}} \exp\left(-\frac{a^2}{2r} - \frac{b^2}{2s}\right) \tag{18}$$

where $p_{AB}(a,b)$ is the two-dimensional normal distribution of $\hat{C}_r(\alpha)$ and $\hat{C}_i(\alpha)$. Transforming to polar coordinates,

$$A = M\cos\psi\,; \qquad B = M\sin\psi \tag{19}$$

we obtain the required distributions for the modulus M and the phase ψ:

$$p_M(m) \; = \; \int_0^{2\pi} p_{M\psi}(m,\varphi)m\,d\varphi = \frac{m}{\sqrt{rs}} \exp\left(-\frac{m^2}{4r} - \frac{m^2}{4s}\right) \tag{20}$$

$$p_\psi(\varphi) \; = \; \int_0^{+\infty} p_{M\psi}(m,\varphi)m\,dm = \frac{1}{2\pi}\frac{\sqrt{rs}}{s\cos^2\varphi + r\sin^2\varphi} \tag{21}$$

These formulas show that $p_M(m)$ is the Hoyt distribution [3] and that the phase is not uniformly distributed between 0 and 2π. Nevertheless, outside the specular reflection zone and if L goes to infinity, $p_M(m)$ is reduced to the Rayleigh distribution and the phase is uniformly distributed.

5. Results

Figure 2 gives the incoherent intensity for a Gaussian random profile having a modulation length $L = 24\lambda$, a rms height $h = 3\lambda/100$ and a correlation length $l_c = 2\lambda$. We can note the zero value of $I_f(\theta)$ in the specular direction ($\theta = \theta_i = 30°$). Outside the specular zone, the comparison with results obtained by the C method [10] is good. The dashed curve and the solid curve show the values obtained by (15) and by the C method.

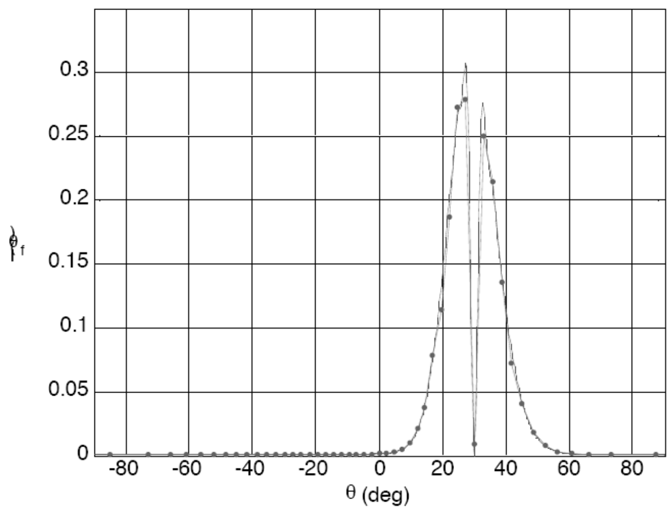

Figure 2: Incoherent intensity for a Gaussian random profile.

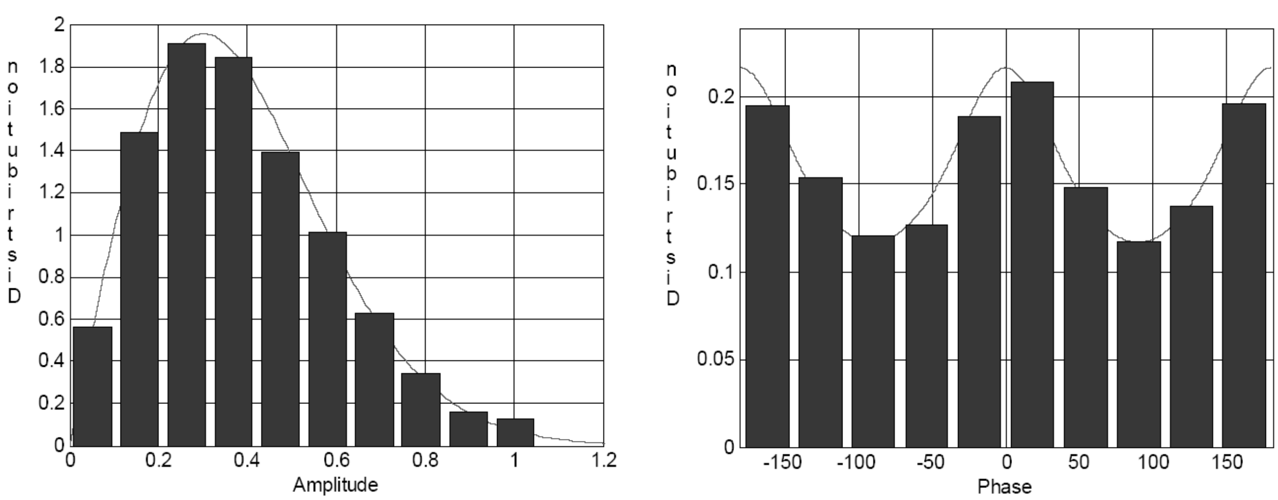

Figure 3: Amplitude and phase distributions.

Figure 3 show the values of the Hoyst distribution and the phase distribution (given by (20) and (21), respectively) for an observation angle $\theta = 10°$. The comparison with the normalized histogram obtained by a Monte-Carlo simulation with 10000 surface realizations is good.

6. Conclusion

We have derived the statistical distribution in the far field zone from the first-order small perturbation method in the particular case of perfectly conducting 1D random rough surfaces illuminated by an $E_{//}$ polarized monochromatic plane wave. We have shown that the real part and the imaginary part of scattering amplitudes are uncorrelated Gaussian stochastic variables with zero mean values and unequal variances. The probability density for the amplitude is given by the Hoyt distribution and the phase is not uniformly distributed between 0 and 2π. Comparisons with statistical observation over 10000 surfaces confirm the result. This approach can be extended to dielectric random rough surfaces illuminated by a polarized plane wave $E_{//}$ or $H_{//}$. The generalization of these results to slightly rough surface with an arbitrary statistical height distribution with an arbitrary correlation function is in progress.

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