An Effective Inversion Method Based on the Padé via Lanczos Process

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Abstract—In this paper we present a nonlinear effective inversion method based on the Padé Via Lanczos process (PVL process). The method finds so-called effective medium parameters of some inhomogeneous object by minimizing an objective function which describes the discrepancy between the scattered field produced by an inhomogeneous object and the scattered field produced by a homogeneous one. This minimization procedure can be carried out by inspection, since the scattered field produced by homogeneous objects can be computed very efficiently using the PVL process. The constant medium parameters of the homogeneous object for which the objective function is minimum are the effective medium parameters we are looking for. A number of numerical experiments are presented in which we illustrate the performance of the method.

1. Introduction

We consider a two-dimensional configuration that is invariant in the z-direction. The position vector in the transverse xy-plane is denoted by x. An object, with known support $\mathbb{S}^{\mathrm{obj}}$, is located in vacuum and is characterized by a conductivity $\sigma(\mathrm{x})$ and a permittivity $\varepsilon(\mathrm{x})$. The object is illuminated by E-polarized waves which are generated by a line source of the form

$$J_z^{\mathrm{ext}}(\mathrm{x},\omega) = f(\omega)\delta(\mathrm{x}-\mathrm{x}^{\mathrm{src}}), \tag{1}$$

where $f(\omega)$ is the source signature, and the delta function on the right-hand side is the Dirac distribution operative at $\mathrm{x} = \mathrm{x}^{\mathrm{src}}$. The source is located outside the object ($\mathrm{x}^{\mathrm{src}} \notin \mathbb{S}^{\mathrm{obj}}$), and the incident electric field strength generated by the line source is given by

$$E_z^{\mathrm{inc}}(\mathrm{x},\omega) = \gamma_s H_0^{(1)}(k_0|\mathrm{x}-\mathrm{x}^{\mathrm{src}}|), \tag{2}$$

where $H_0^{(1)}$ is the zero-order Hankel function of the first kind, $\gamma_s = i\omega\mu_0 f(\omega)/4$, and k_0 is the wave number of vacuum.

The total electric field strength is measured at some receiver location $\mathrm{x}^{\mathrm{rec}} \in \mathbb{S}^{\mathrm{obj}}$ and since the incident electric field is known, the scattered electric field strength at the receiver location is known as well. We denote this scattered field by E_z^{sc}. In what follows we assume that this field does not vanish at the receiver location.

The full inversion problem consists of retrieving the conductivity $\sigma(\mathrm{x})$ and permittivity $\varepsilon(\mathrm{x})$ of the object from the measured electric field strength. In our effective inversion method, however, we follow a different approach. We act as if the object is homogeneous and try to find position-independent medium parameters for which the scattered field at the receiver location matches the true scattered field using a well-defined objective function.

Let us be more precise. Introducing the contrast coefficient of the homogeneous object as

$$\zeta(\omega) = \tilde{\varepsilon}_{\mathrm{r}} - 1 + i\frac{\tilde{\sigma}}{\omega\varepsilon_0} \tag{3}$$

where $\tilde{\varepsilon}$ and $\tilde{\sigma}$ are position-independent, we have for the scattered field at the receiver location the integral representation

$$\tilde{E}_z^{\mathrm{sc}}(\mathrm{x}^{\mathrm{rec}},\omega) = \frac{ik_0^2}{4}\zeta(\omega)\int_{\mathrm{x}'\in\mathbb{S}^{\mathrm{obj}}} H_0^{(1)}(k_0|\mathrm{x}^{\mathrm{rec}}-\mathrm{x}'|)\tilde{E}_z(\mathrm{x}',\omega)dA. \tag{4}$$

This so-called data equation relates the scattered field at the receiver location to the contrast coefficient and the total electric field inside the object. This total field is unknown, but we do know that it satisfies the object equation

$$\tilde{E}_z(\mathrm{x},\omega) - \frac{ik_0^2}{4}\zeta(\omega)\int_{\mathrm{x}'\in\mathbb{S}^{\mathrm{obj}}} H_0^{(1)}(k_0|\mathrm{x}-\mathrm{x}'|)\tilde{E}_z(\mathrm{x}',\omega)dA = E_z^{\mathrm{inc}}(\mathrm{x},\omega) \quad \text{with} \quad \mathrm{x}\in\mathbb{S}^{\mathrm{obj}} \tag{5}$$

This object equation is an integral equation of the second kind for the total electric field strength $\tilde{E}_z$ for a given value of the contrast coefficient.

Discretizing the object and data equation on a uniform grid using square discretization cells with side lengths δ is standard and we do not discuss it in this paper. We only give the final forms of the discretized data and object equations, and refer to [1] for details on the discretization process.

After the spatial discretization procedure we obtain the discretized data equation

$$u^{\mathrm{sc}}(\zeta) = \gamma_{\mathrm{r}}\zeta \mathbf{r}^T \mathbf{u}, \tag{6}$$

where $\gamma_{\mathrm{r}} = i(k_0\delta)^2/4$, $\mathbf{r}$ is a receiver vector, and $\mathbf{u}$ is a vector containing the expansion coefficients of the total electric field inside the object. Furthermore, the discretized object equation for the homogeneous object is given by

$$(I - \zeta G)\mathbf{u} = \mathbf{u}^{\mathrm{inc}} \tag{7}$$

where I is the identity matrix, and matrix G is a square and symmetric (but not a Hermitean) matrix with (scaled) Green's function values as its entries. Since matrix G results from a discretization of a convolution operator on a uniform grid, we can compute its action on a vector very efficiently using the Fast Fourier Transform (FFT). Finally, the vector $\mathbf{u}^{\mathrm{inc}}$ is a vector consisting of incident electric field strength values. This vector can be written in the form $\mathbf{u}^{\mathrm{inc}} = \gamma_s \mathbf{s}$ where $\mathbf{s}$ is such that $\mathbf{s} = \mathbf{r}$ if the source and receiver locations coincide. Using the latter form for the incident field vector in the discretized object equation, solving this equation for the total field $\mathbf{u}$, and substituting the result in the discretized data equation, we arrive at

$$u^{\mathrm{sc}}(\zeta) = \gamma\zeta \mathbf{r}^T (I - \zeta G)^{-1}\mathbf{s}, \tag{8}$$

with $\gamma = \gamma_{\mathrm{r}}\gamma_{\mathrm{s}}$. If we compute the scattered field u^{sc} equation (8) directly, we have to solve a forward problem for each new value of ζ. Such a procedure can be computationally intensive and it turns out that it can be avoided using the Padé Via Lanczos (PVL) process. We briefly describe this process in the next section.

2. The Padé via Lanczos Process

We first define our domain of interest. Let $\tilde{\varepsilon}_{\mathrm{r;max}}$ and $\tilde{\sigma}_{\mathrm{max}}$ be a priori given upper bounds for the constant medium parameters. Then our domain of interest is defined as

$$\mathbb{T} = \{\zeta \in \mathbb{C}; 0 \leq \mathrm{Re}(\zeta) \leq \tilde{\varepsilon}_{\mathrm{r};max} - 1, 0 \leq \mathrm{Im}(\zeta) \leq \tilde{\sigma}_{\mathrm{max}}/(\omega\varepsilon_0)\},$$

since we require that $\tilde{\varepsilon}_{\mathrm{r}} \geq 1$ and $\tilde{\sigma} \geq 0$. We now compute $[k-1/k]$-Padé approximations for the scattered field u^{sc} around an expansion point $\zeta_0 \in \mathbb{T}$ by performing k iterations of the two-sided Lanczos algorithm (see [2]). Matrix factorization is required for any nonzero expansion point and computing such a factorization is expensive (although it has to be computed only once). However, no such factorization is needed if we take $\zeta_0 = 0$ as an expansion point. Only matrix-vector products with matrix G are required in this case and, as we have mentioned above, such products can be computed efficiently using FFT. We therefore construct $[k-1/k]$-Padé approximations for the scattered field u^{sc} around the expansion point $\zeta_0 = 0$ by performing k iterations of the two-sided Lanczos algorithm using the source and receiver vectors s and r as starting vectors. We denote the resulting Padé approximation by u_k^{sc}. The crux of the matter is that to evaluate this approximation for each $\zeta \in \mathbb{T}$, we need to solve a k-by-k tridiagonal system and k is typically much smaller than the order of the original discretized object equation. Assuming now that k is such that essentially

$$u_k^{\mathrm{sc}}(\zeta) = u^{\mathrm{sc}}(\zeta) \quad \text{for all} \quad \zeta \in \mathbb{T},$$

we can conclude that we have an efficient way of evaluating the scattered field for all ζ-values of interest.

3. The Effective Medium Parameters

The effective medium parameters follow from minimizing an objective function defined over the domain of interest. More precisely, the effective medium parameters are defined as those parameters for which the objective function

$$F_1(\zeta) = \frac{|E_z^{\mathrm{sc}} - u_k^{\mathrm{sc}}|^2}{|E_z^{\mathrm{sc}}|^2} \tag{9}$$

attains a minimum in our domain of interest $\mathbb{T}$. If multiple frequency data $E_z^{\mathrm{sc}}(\omega_1)$, $E_z^{\mathrm{sc}}(\omega_2)$, $\cdots$, $E_z^{\mathrm{sc}}(\omega_N)$ is

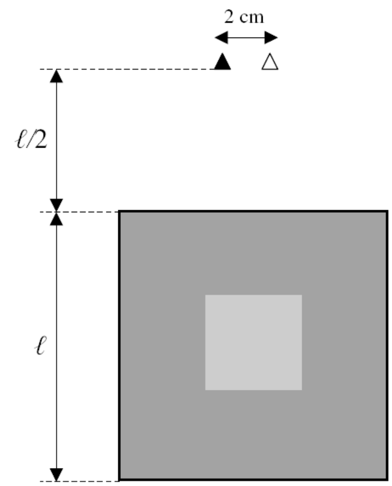

Figure 1: Two-dimensional test configuration.

available, we look for those medium parameters for which the multiple frequency objective function

$$F_N(\zeta) = \sum_{n=1}^{N} \omega_n \frac{|E_z^{\rm sc}(\omega_n) - u_k^{\rm sc}(\omega_n)|^2}{|E_z^{\rm sc}(\omega_n)|^2} \tag{10}$$

is minimum. In the above equation, the weights ω_n satisfy $\sum_{n=1}^{N} \omega_n = 1$. Notice that in the multiple frequency case we have to apply the PVL process for each frequency separately. Moreover, for multiple frequencies the domain of interest on which all the PVL approximations match the true scattered field due to a homogeneous object is taken to be the domain $\mathbb{T}$ which corresponds to the lowest frequency of operation. Minimizing the objective functions can be carried out by inspection since we have a very efficient way of computing the scattered fields $u_k^{\rm sc}(\omega_n)$. Finally, we mention that we cannot guarantee that the effective medium parameters are unique. The objective function may have multiple minima on the domain of interest and each minimum gives a set of effective medium parameters for the object. However, usually we can overcome the nonuniqueness of the effective medium parameters by including more a priori information, or by performing additional experiments at different frequencies while keeping the source/receiver unit fixed.

4. Numerical Results

We illustrate our effective inversion approach using the two-dimensional configuration shown in Figure 1. A square block with side lengths ℓ is located in a vacuum domain. The block has an inner and an outer part and each part has its own constant medium parameters. Specifically, the outer part has a conductivity σ_1 and a relative permittivity $\varepsilon_{\rm r;1}$, the inner part a conductivity σ_2 and a relative permittivity $\varepsilon_{\rm r;2}$. Obviously, the block is homogeneous if σ_1=σ_2 and $\varepsilon_{\rm r;1} = \varepsilon_{\rm r;2}$. Finally, the source/receiver unit is located a distance $\ell/2$ above the object and the source and the receiver are located 2 cm apart.

In our first example, we operate at a frequency of 36 MHz, and take $\ell = \lambda_{36}$, where λ_{36} is the free-space wavelength corresponding to the operating frequency of 36 MHz. The block is homogeneous with $\sigma_1 = \sigma_2 = 7.5\,$mS/m and $\varepsilon_{\rm r;1} = \varepsilon_{\rm r;2} = 5$. For the maximum conductivity and maximum relative permittivity we take $\tilde{\sigma}_{\rm max} = 10\,$mS/m and $\tilde{\varepsilon}_{\rm r;max} = 6$, respectively. The domain of interest is discrerized on a 50-by-50 grid (leading to 2500 forward problems solved by PVL in less than a second on a notebook with a 1.6 GHz Pentium M processor) and the objective function F_1 on this domain of interest is shown in Figure 2 (left). We observe that the true conductivity and permittivity of the object are recovered. However, a number of additional minima are present near the $\tilde{\varepsilon}_{\rm r}$-axis. To remove these minima we add two more frequency measurements, namely, one at a frequency of $f = 30\,$MHz and one at $f = 42\,$MHz. The objective function F_3 for these two frequencies and the frequency of 36 MHz is shown in Figure 2 (right), where we have taken $\omega_n = 1/3$ for $n = 1, 2, 3$. Clearly, the multiple minima have disappeared and a single minimum remains. In addition to using multiple frequency data, we could also change the source and receiver locations. This latter option is not considered in this paper, however.

We now apply our effective inversion method to inhomogeneous blocks. Two blocks of different sizes will be

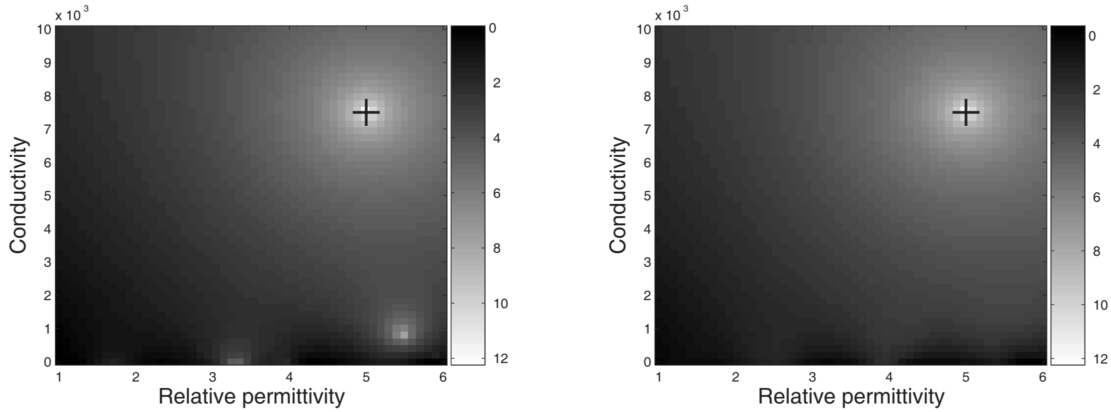

Figure 2: Base 10 logarithm of F_1 (left) and base 10 logarithm of F_3 (right) on the domain of interest.

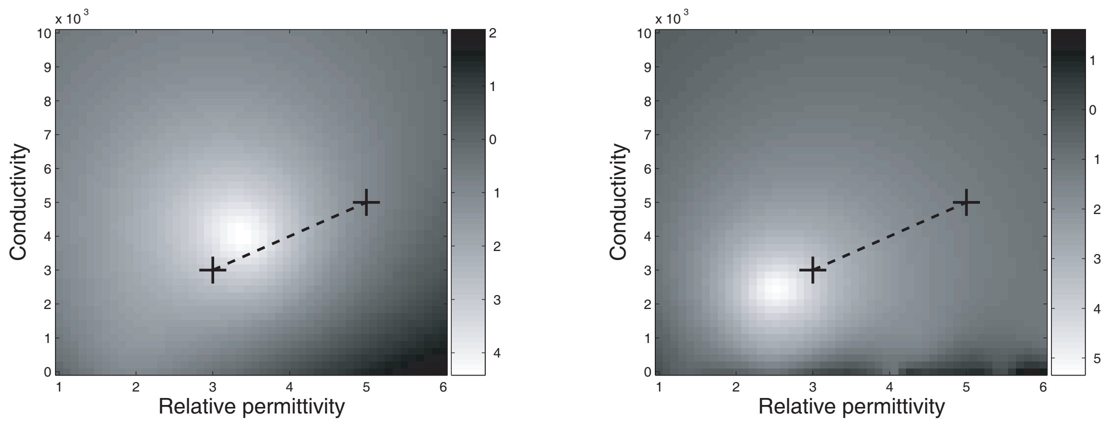

Figure 3: Base 10 logarithm of F_3 for the $\lambda_{36}/4$-block (left) and the λ_{36}-block (right).

considered. The first block has a side length $\ell = \lambda_{36}/4$ and the second one a side length of $\ell = \lambda_{36}$. The outer part of the two blocks has a conductivity $\sigma_1 = 3.0\,\mathrm{mS/m}$ and a relative permittivity $\varepsilon_{\mathrm{r};1} = 3$, while the medium parameters of the inner part are $\sigma_2 = 5.0\,\mathrm{mS/m}$ and $\varepsilon_{\mathrm{r};2} = 5$. For both blocks the area of the inner part is 50% of the total area of the block. Using the same three frequencies as in the previous examples, we obtain the objective functions as shown in Figure 3. The minimum for the $\lambda_{36}/4$-block is located at an acceptable location in the domain of interest, but for the large block the effective medium parameters are smaller than the smallest medium parameters of the block. This result is unexpected. We therefore carried out an additional number of experiments and all these experiments indicate that for inhomogeneous objects it all depends on the size of the object and the sizes of the perturbations with respect to a constant contrast function. This latter function may be large, but the perturbations cannot be "too large". Finding a condition that tells us for which contrast perturbations the proposed method gives reliable results is a topic we are presently investigating. In addition, we want to know how this condition changes if the data is perturbed (by noise, for example) given the magnitude of the data perturbations.

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