Mathematical Modeling of Nonlinear Waves and Oscillations in Gyromagnetic Structures by Bifurcation Theory Methods

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Abstract—The vector field bifurcation approach and its numerical implementation for the rigorous mathematical simulation of nonlinear phenomena in microwave and mm-wave ferrite or composite semiconductor/ferrite devices are developed. The bifurcation points of nonlinear Maxwell's operator for the three-dimensional boundary problems, stated and solved rigorously (i. e., considering the full Maxwell's equations together with the nonlinear equations of motion for magnetization in ferrites and transport carriers in semiconductors) are analyzed using numerical methods. The electromagnetic field is represented as decomposed into a series of weakly nonlinear wave fields. The solutions of a linearized Maxwell's operator matrix equation are determined. The propagation constants of weakly nonlinear waves in waveguiding structures (WGS) or eigenfrequencies of weakly nonlinear oscillations in resonator structures (RS) are found. Using the bifurcation dynamics of Maxwell's equations the nonlinear wave interactions in the strongly nonlinear planar ferrite insert, loaded into strip-slot RS, are analyzed (from the harmonic frequency terms at the 'soft' non-linear stage into the region of 'hard' nonlinearity). The nonlinear propagation of electromagnetic waves in the strip-slot ferrite RS are modeled. The nonlinear wave phenomena, including the parametric excitation of oscillations and the wave instability process are investigated taking into account constrained geometry WGS and RS.

1. Introduction

The research of bifurcations in nonlinear dynamical systems with distributed parameters, described by nonlinear differential equations in partial derivatives, involves serious mathematical difficulties. As for distributed systems a characteristic determinant is an analogue of frequency characteristics, that's why it is possible to analyze distributed self-sustained oscillation systems using the linearization method combined with the characteristic determinant analysis (at first it was shown in [1] for the one-dimensional case). Hitherto the bifurcation analysis was used to investigate nonlinear dynamical systems with lumped parameters, described by nonlinear ordinary differential equations (ODEs). When the ordinary differential equation is of second order a qualitative analysis is possible on the two-dimensional phase surface [2]. The linearization method in combination with the frequency-domain analysis is used for the analysis of self-sustained oscillating systems and automatic control systems [2]. Determining the solutions of nonlinear differential equations in fixed points using numerical computation is a very complicated problem even for ODEs, because at the branching points qualitative modifications of solutions can happen due to variation of parameters.

The behavior caused by the instability of waves and oscillations in nonlinear or parametric systems, containing nonlinear magnetic or semiconductor media, is complex [3]. The physical theories of the instability of magnetostatic or spin waves were developed using the approximate analysis of the equation of motion of the magnetization vector in ferromagnet for one-dimensional structures only [4, 5]. The analysis of the transition region from the stable regime to the onset of labile oscillating mode caused the instability is the most complicated problem. This analysis can only be based on the solutions of full nonlinear Maxwell's equations, complemented by the nonlinear equations of motion of the magnetization vector in a ferromagnet [3]. The goal of this paper is to develop a new approach based on the bifurcation theory [6, 7] for accurate electromagnetic modeling of nonlinear wave phenomena in gyromagnetic or semiconductor waves in waveguiding structures (WGS) or resonator structures (RS) using a numerical approach for the analysis of the linearized matrix equation and bifurcation points of the nonlinear Maxwell's operator. It opens up new prospects of bifurcation analysis and rigorous mathematical modeling of strongly nonlinear electrodynamical systems using the bifurcation dynamics of Maxwell's equations.

2. The Numerical Method of Linearization of Nonlinear Maxwell's Operator in Combination with the Analysis of the Characteristic Determinant

The numerical method to determine the propagation constants of weakly nonlinear waves in WGS (or eigenfrequencies of weakly nonlinear oscillations in RS) loaded with strongly nonlinear gyromagnetic or semiconductor boundary media consists in the following.

The three-dimensional boundary problems, stated rigorously (i. e., considering the full Maxwell's equations with the nonlinear equation of motion for magnetization in ferrites or the equation of transport carriers in semiconductors, with boundary conditions following from conditions of non-asymptotic radiation) was reduced to the boundary problem for a system of nonlinear DEs together with the system of the nonlinear algebraic equations using the cross-sections method in [8, 9].

The system of nonlinear DEs together with the system of nonlinear algebraic equations [8, 9] is represented in a symbolic form, as:

$$\frac{dy_i}{dz} = F_i(y_1, y_2, \ldots, y_n), \qquad \Psi_j(y_1, y_2, \ldots, y_n) = 0, \tag{1}$$

where $i = 1, 2, \ldots, m$; $j = m+1, m+2, \ldots, n$; $y_i = y_i(z)$ are unknown functions of the longitudinal coordinate z compiled on the functions $a_n^t(\omega_m)$, $b_n^t(\omega_m)$, $a_n^z(\omega_m)$, $b_n^z(\omega_m)$, given in references [8, 9].

Let $y_i = 0$ $(i = 1, 2, \ldots, n)$ be the solution of the system (1), satisfying the boundary conditions as given in reference [8, 9]. Then the functions F_i and Ψ_j $(i = 1, 2, \ldots, m; j = m+1, m+2, \ldots, n)$ identically vanish, consequently, the solution $y_i = 0$ $(i = 1, 2, \ldots, n)$ of the system (1) is fixed (stationary) relative to the coordinate variable z.

As the first approximation, reduce the system of nonlinear differential equations (1) to a system of linear differential equations. For this purpose it is necessary to represent functions F_i and Ψ_j by their generalized Taylor's series in the neighborhood of fixed (stationary) points $x_i = 0$, and to take into account the first order partial derivatives. This procedure results a system of linear differential equations:

$$\frac{dy_i}{dz} = \sum_{K=1}^{n} \frac{\partial F_i(0, 0, \ldots, 0)}{\partial y_K} \cdot y_K, \qquad \sum_{K=1}^{n} \frac{\partial \Psi_j(0, 0, \ldots, 0)}{\partial y_K} \cdot y_K = 0, \tag{2}$$

where $i = 1, 2, \ldots, m; j = m+1, m+2, \ldots, n$.

Let us represent the system of differential equations (2) in expanded form:

$$\begin{cases} a_{11}(z) \cdot y_1 + a_{12}(z) \cdot y_2 + \ldots + a_{1n}(z) \cdot y_n = y_1', \\ a_{21}(z) \cdot y_1 + a_{22}(z) \cdot y_2 + \ldots + a_{2n}(z) \cdot y_n = y_2', \\ \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \\ a_{m1}(z) \cdot y_1 + a_{m2}(z) \cdot y_2 + \ldots + a_{mn}(z) \cdot y_n = y_m', \\ a_{m+1,1}(z) \cdot y_1 + a_{m+1,2}(z) \cdot y_2 + \ldots + a_{m+1,n}(z) \cdot y_n = 0, \\ a_{m+2,1}(z) \cdot y_1 + a_{m+2,2}(z) \cdot y_2 + \ldots + a_{m+2,n}(z) \cdot y_n = 0, \\ \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \\ a_{n1}(z) \cdot y_1 + a_{n2}(z) \cdot y_2 + \ldots + a_{nn}(z) \cdot y_n = 0, \end{cases} \tag{3}$$

where the coefficients $a_{ij}(z)$ $(i, j = 1, 2, \ldots, n)$ compiled on the partial derivatives from (2). The system of equations (3) can be represented in matrix form as:

$$A \cdot y = \frac{dy}{dz} \tag{4}$$

where y is the vector with components $y_1, y_2, \ldots, y_m$; $\dfrac{dy}{dz}$ is the vector with components $y_1', y_2', \ldots, y_m'$; $A = A_{11} - A_{12} \cdot A_{22}^{-1} \cdot A_{21}$,

$$A_{11} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} a_{1m+1} & a_{1m+2} & \ldots & a_{1n} \\ a_{2m+1} & a_{2m+2} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m,m+1} & a_{m,m+2} & \ldots & a_{mn} \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} a_{m+1,1} & a_{m+1,2} & \ldots & a_{m+1,m} \\ a_{m+2,1} & a_{m+2,2} & \ldots & a_{m+2,m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{pmatrix}, \qquad A_{22} = \begin{pmatrix} a_{m+1,m+1} & a_{m+1,m+2} & \ldots & a_{m+1,n} \\ a_{m+2,m+1} & a_{m+2,m+2} & \ldots & a_{m+2,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix},$$

We find the partial solutions of the system of equation (4) in the form:

$$y = \alpha \cdot e^{\lambda \cdot z}, \tag{5}$$

where α is the vector with components $\alpha_1, \alpha_2, \ldots, \alpha_m$. Substituting (5) into (4), we obtain the following eigenvalue matrix equation:

$$A \cdot \alpha = \lambda \cdot \alpha, \tag{6}$$

where λ and α are correspondingly the eigenvalues and eigenvectors of matrix A. Using a numerical method (for example, the QR-algorithm) to solve the matrix equation (6) the eigenvalues λ_m and eigenvectors α of A can be determined.

The solutions (5) of the linearized Maxwell's operator (6) are treated as weakly nonlinear waves. The electromagnetic field in WGS is decomposed into a series of weakly nonlinear wave fields. The eigenvalues λ_m of matrix A are the propagation constants of the weakly nonlinear waves in WGS (or the eigenfrequencies of weakly nonlinear oscillations in RS). The components of the eigenvectors α of matrix A are the transverse and longitudinal components of weakly nonlinear waves .

The computational algorithm, using the linearization of nonlinear Maxwell's operator and the decomposition into a series of weakly nonlinear wave fields, is more complex than those for the propagation constants and fields of eigenwaves of WGS, filled with a linear medium. But the convergence of this algorithm and its stability for rounding errors is better. It permits to solve the threedimensional diffraction boundary problems for WGS or RS loaded with strongly nonlinear gyromagnetic or semiconductor insertions having sizes comparable to the wavelength. This is important for CAD of prospective ferrite or composite semiconductor/ferrite devices at microwave or mm-waves.

3. Numerical Simulation of the Parametric Excitation of Oscillations in Nonlinear Gyromagnetic Structure Using Bifurcation Points

The rigorous mathematical modeling of parametric oscillations in strip-slot RS loaded with a planar magnetized ferrite (Fig. 1) is based on solving the nonlinear diffraction boundary problem by the crosssections method of [8], using the decomposition algorithm on nonlinear autonomous blocks [10].

For the computational algorithm the transverse and longitudinal components of weakly nonlinear waves are used. It results a stable and computationally efficient algorithm for computing the instability of waves or oscillations in WGS or RS containing strongly nonlinear gyromagnetic media.

There are two incident electromagnetic waves: the signal wave of frequency ω_1 and the pumping wave of frequency ω_2 are incident on the input cross-sections S_1 of RS (Fig. 1). The waves are the fundamental and higher-order modes of strip-slot WGS, having magnitudes $C^+_{n(\alpha)}(\omega_1)$ and/or $C^+_{n(\alpha)}(\omega_2)$, where α is the index of the cross-sections, n are the indices of eigenwaves of strip-slot WGS [8, 9].

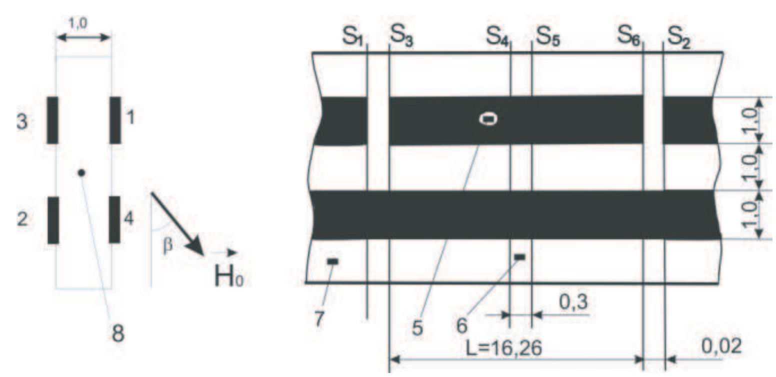

Figure 1: Resonator structure with nonlinear ferrite insert: 1, 2, 3, 4—coupled strips of strip-slot WGS; 5— strip-slot resonator; 6—planar magnetized ferrite insert ($\varepsilon = 9$, $H_0 = 278$ A/mm; $M_0/\mu_0 = 160$ A/mm; $\omega_r = 3{*}10^9$ Hz; $\beta = 45°$); 7—dielectric substrate ($\varepsilon = 9$; $\mu = 1$); 8—point of field observation; $f_1 = 5$ Hz, $f_2 = 10$ GHz; all sizes are in mm.

The instability of parametric excitation process of oscillations in ferrite RS depending on the bifurcation parameters is simulated using the numerical method of bifurcation points analysis, developed by us [11]. The results of computing of the instability regions for parametric excitation of oscillations in ferrite RS by the incident pumping wave, depending on the magnitude $C^+_{2(1)}(\omega_2)$ and the normalized frequency (the signal frequency ω_1 with respect to the pumping frequency ω_2) are shown in Fig. 2. The onset and the breakdown of parametric oscillations caused the wave instability in nonlinear ferrite structure in the neighborhood of bifurcation parameters were simulated into the region of 'hard' nonlinearity taking into account constrained geometries RS, and it is represented in Fig. 2.

It follows from the results of the mathematical modeling that the unstable regions for parametric excitation of oscillations in ferrite RS are near the values of the eigenfrequencies of fundamental and higher-order modes of oscillations of the strip-slot line resonator: $\omega_1 = m\omega_2/2$, $m = 1, 2, 3, \ldots$ The threshold magnitude $C^+_{2(1)}(\omega_2)$ is rising steeply as m increases. The minimum threshold of $C^+_{2(1)}(\omega_2)$ is given by $\omega_1 = \omega_2/2$.

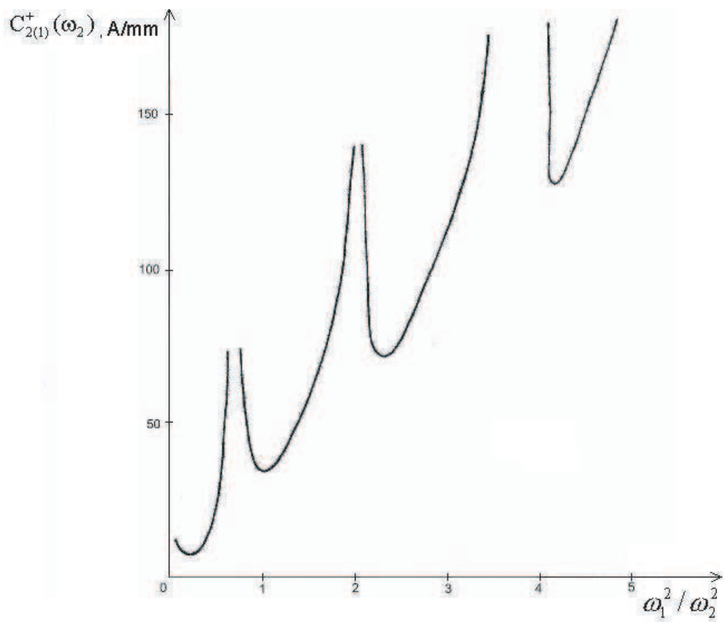

Figure 2: Instability regions for parametric excitation of oscillations in nonlinear ferrite RS, depending on bifurcation parameters: $C^+_{2(1)}(\omega_2)$—magnitude of incident pumping wave; ω_1—eigenfrequency of fundamental modes of oscillations the strip-slot line resonator (length of the resonator = half-wave for signal wave at $f_1 =$ 5 GHz); ω_2 —frequency of pumping wave.

4. Conclusion

Using the achievements of modern mathematics in the area of vector field bifurcation theory opens new possibilities for computer analysis of the onset of nonlinear waves in WGS with bounded gyromagnetic media having a strong nonlinearity. This approach has a high likelihood of success in investigating nonlinear phenomena in new microwave/millimeter-wave ferrite devices [12] for frequency multiplexing/filtering, limiters, noise rejectors, signal-noise ratio enhancers, and pulse compressing devices.

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