On the Intermittency of the Light Propagation in Disordered Optical Materials

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Abstract—We consider propagation of light through an ensemble of $N \gg 1$ statistically independent optical fibers of length L whose refraction coefficient is a random function of length. We introduce the generalized transmission coefficient $|t(k, L)|^p$ for energy k^2 and study its quenched and annealed Lyapunov exponents. For small disorder we calculate the Lyapunov exponents in asymptotic form.

1. Introduction

The idea of intermittency was originally proposed in the study of turbulent flow [1] and has become widespread in statistical particle physics. Intermittency means random deviations from smooth and regular behavior. To illustrate it, we consider a bundle of N, $N \gg 1$, statistically equidistributed independent optical fibers of a fixed length L whose refractive index changes randomly along the length of the fiber. If one face of the bundle is illuminated then, due to reflection of the light and its localization in the fibers, one might expect that the outlet of the bundle will be uniformly dark. However, because of strong statistical fluctuations of the transparency (that is a typical manifestation of the intermittency), the exit of the bundle will look like a dark sky with sparse bright stars. This model was proposed by I. M. Lifshits [2] to explain high irregularity of the light distribution after propagation through a thick layer of a disordered optical material. Propagation of light in each fiber is described by the equation

$$-\psi'' + \sigma V_j(x)\psi = k^2\psi, \quad j = 1, 2, \ldots, N, \tag{1}$$

where $V_j(x)$ are homogeneous random potentials equal zero outside the fibers and constant σ characterizes strength of the disorder.

Equation 1 has scattering solutions

$$\psi_{k,j}(x) = \begin{cases} e^{ikx} + r_j(k)\, e^{-ikx}, & x < 0, \\ t_j(k)\, e^{ikx}, & x > L, \end{cases} \tag{2}$$

where $t_j(k)$ and $r_j(k)$ are random complex transmission and reflection coefficients, respectively, such that $|t_j(k)|^2 + |r_j(k)|^2 = 1$. We also introduce the empirical mean $\frac{1}{N}\sum_{j=1}^{N} |t_j(k)|^2$ for the transmitted energy provided the energy density of the incident wave equals one for each waveguide, and for fixed L and $N \to \infty$

$$\frac{1}{N}\sum_{j=1}^{N} |t_j(k)|^2 \xrightarrow{a.s.} \langle |t(k, L)|^2 \rangle, \tag{3}$$

where a.s. means almost surely (with probability one). Expressions $|t(k, L)|^p$ and $\langle |t(k, L)|^p \rangle$ are decreasing exponentially as $L \to \infty$ whose logarithmic rate of decay we call the quenched and annealed (moment) transmission Lyapunov exponents, respectively,

$$\gamma_q^T(k, p) = \lim_{L\to\infty} \frac{\ln |t(k, L)|^p}{L} = p \lim_{L\to\infty} \frac{\ln |t(k, L)|}{L} = p\gamma^T(k), \tag{4}$$

$$\mu_a^T(k, p) = \lim_{L\to\infty} \frac{\ln\langle |t(k, L)|^p \rangle}{L}. \tag{5}$$

Using this notation we can quantitatively characterize intermittency: after propagation through the fiber bundle light exhibits intermittency if

$$|\mu_a^T(k, 2)| < |\gamma_q^T(k, 2)|. \tag{6}$$

The stronger inequality (6) is, the more intermittent is the distribution of energy on the exit of the fiber bundle.

2. Analytical Tools

The study of equation 1 with representative potential $V(x)$ is based on the phase-amplitude formalism. Let $\psi_k^{(i)}(x), i = 1, 2$, be the fundamental set of solutions of (1) with initial values $\psi_k^{(1)}(0) = 1$, $\frac{d}{dx}\psi_k^{(1)}(0) = 0$, $\psi_k^{(2)}(0) = 0$, $\frac{d}{dx}\psi_k^{(2)}(0) = 1$. The matrix

$$M_k([0, L]) = \begin{pmatrix} \psi_k^{(1)}(L) & k\psi_k^{(2)}(L) \\ \frac{1}{k}\frac{d}{dx}\psi_k^{(1)}(L) & \frac{d}{dx}\psi_k^{(2)}(L) \end{pmatrix} \tag{7}$$

is the propagator of (1) whose determinant equals one.

For the general solution of (1) we put

$$\psi_k(x) = r_k(x)\sin\theta_k(x), \quad \frac{d\psi_k(x)}{dx} = kr_k(x)\cos\theta_k(x). \tag{8}$$

Then for θ_k and $\ln r_k$ we obtain the following system [2], [3]

$$\frac{d\theta_k(x)}{dx} = k - \frac{\sigma V(x)\sin^2\theta_k}{k}, \tag{9}$$

$$\frac{d\ln r_k(x)}{dx} = \frac{1}{2k}\sin 2\theta_k(x)V(x). \tag{10}$$

In most cases of interest [2], [3], the phase $\theta_k(x) \in [0, \pi)$ represents either a Markov process with generator $\mathcal{L}$ (white noise potential) or a component of a multidimensional Markov process (the Kronig-Penny model). To illustrate intermittent behavior of light distribution, we use the simplest case when the potential $V(x) = \dot{b}(x)$ is the white noise (the derivative of the Brownian motion $b(x)$).

Equations 9–10 are understood as Itô's stochastic differential equations with Stratonovich corrections. In our case, the generator of the diffusion process (9) has the form [4]

$$(\mathcal{L}f)(\theta) = \frac{B^2(\theta)}{2}\frac{d^2 f}{d\theta^2} + \left(A(\theta) + \frac{(BB')(\theta)}{2}\right)\frac{df}{d\theta}, \tag{11}$$

where $A(\theta) = k$, $B(\theta) = -\dfrac{\sigma\sin^2\theta}{k}$. Similarly,

$$d(\ln r(x)) = \left(\alpha(\theta(x)) + \frac{1}{2}\beta B(\theta(x))\right)dx + \beta(\theta(x)) \cdot db(x) \tag{12}$$

with $\alpha = 0$ and $\beta(\theta) = \dfrac{\sigma\sin 2\theta}{2k}$. Hence,

$$r^p(x) = e^{\int_0^x D(\theta)\cdot db(z) + \int_0^x C(\theta)\,dz}, \tag{13}$$

where $D(\theta) = p\beta(\theta(z))$ and $C(\theta) = p(\alpha + \frac{1}{2}\beta B)(\theta(z))\,dz$. If $u_p(x,\theta) = \langle r^p(x)|\theta(0) = \theta\rangle$ is the expectation of $r^p(x)$, then $u_p(x,\theta)$ satisfies the Feynman-Kac formula which for the white noise potential has the form

$$\frac{\partial u_p}{\partial x} = \frac{\sigma^2\sin^4\theta}{2k^2}\frac{\partial^2 u_p}{\partial\theta^2} + \left(k + \frac{\sigma^2(1-p)\sin^2\theta\sin 2\theta}{2k^2}\right)\frac{\partial u_p}{\partial\theta} + \frac{\sigma^2 p\sin^2\theta\cos\theta(p\cos\theta - \sin\theta)}{2k^2}u_p = \widetilde{\mathcal{L}}_p u_p. \tag{14}$$

Formula (14) allows to calculate the Lyapunov exponent for the amplitude $r(L)$. In the quenched case we have

$$\begin{aligned}\frac{\ln r(L)}{L} &= \frac{1}{L}\int_0^L \frac{1}{2}\beta B(\theta(x))\,dx + \beta(\theta(x))\cdot db(x) \xrightarrow{a.s.} \langle\frac{1}{2}\beta B\rangle_\eta \\ &= -\frac{\sigma^2}{4k^2}\int_0^\pi \eta(\theta)\sin 2\theta\sin^2\theta\,d\theta = \gamma_q(k).\end{aligned} \tag{15}$$

Here $\eta(\theta)d\theta$ is the invariant measure for the phase $\theta(x)$ which satisfies the equation

$$\mathcal{L}^*\eta = \frac{d^2}{d\theta^2}\left(\frac{\sigma^2\sin^4\theta}{2k^2}\eta\right) - \frac{d}{d\theta}\left[\left(k + \frac{\sigma^2\sin^2\theta\sin 2\theta}{2k^2}\right)\eta\right] = 0 \tag{16}$$

that can be solved exactly.

Consider now the moment Lyapunov exponent

$$\mu_a(p) = \lim_{L\to\infty} \frac{\ln\langle r^p(L)\rangle}{L}. \tag{17}$$

According to Perron–Frobenius theorem about positive semigroups, $\mu_a(p)$ equals maximum eigenvalue of the nonsymmetric operator $\widetilde{\mathcal{L}}_p$ (14)

$$\widetilde{\mathcal{L}}_p\psi = \mu_a(p)\psi \tag{18}$$

and the corresponding eigenfunction $\psi(x)$ is strictly positive.

The Lyapunov exponent $\gamma(k)$ of the amplitude $r(L)$ and $\mu(p)$ have the following properties:

(a) $\gamma(k) > 0$. This property leads to the localization theorem for the Hamiltonian $H\psi = -\psi'' + \sigma V(x)\psi = \lambda\psi$ on the whole real axis [2], [3].

(b) For fixed k the annealed Lyapunov exponent is analytic in p and convex.

(c) $\mu(p)$ is symmetric with respect to $p = -1$: $\mu(p) = \mu(-p-2)$ and $\dfrac{d\mu}{dp}(0) = \gamma(k)$. In particular, $\mu(0) = \mu(-2) = 0$ (Fig. 1).

(d) For small disorder constant σ and fixed k $\gamma(k) = \dfrac{\pi\sigma^2\widehat{B}(2k)}{4k^2}(1+o(\sigma))$, where $\widehat{B}(2k)$ is the spectral density of the potential V. For the white noise $\gamma(k) = \dfrac{\sigma^2}{8k^2}(1+o(\sigma))$ and $\mu_a(p) \approx \dfrac{1}{2}p(p+2)\sigma^2\gamma(k)$ as $\sigma \to 0$.

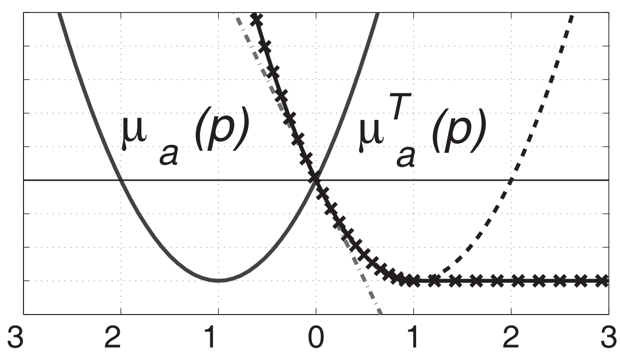

Figure 1: Graphs of the annealed moment Lyapunov exponent $\mu_a(p)$ (solid line) and transmission Lyapunov exponent $\mu_a^T(p)$ (crossed line) for fixed k and small σ.

The energy transmission coefficient can be calculated through the matrix $M_k([0,L])$ (7) as follows

$$|t(k,L)|^2 = \frac{4}{2 + \|M_k([0,L])\|^2}, \tag{19}$$

where the norm is understood as the sum of the squares of matrix's entries. Then $\|M_k([0,L])\|^2 = [r_k^{(1)}(L)]^2 + [r_k^{(2)}(L)]^2$. From asymptotic behavior of the amplitudes $\ln r_k^{(i)}(L) \approx \gamma(k)L, \quad i = 1,2$, with probability one as $L \to \infty$ we conclude that $\ln\|M_k([0,L])\| \approx \gamma(k)L$. Therefore,

$$\frac{\ln|t(k,L)|}{L} = \frac{1}{L}\ln\sqrt{\frac{4}{2 + \|M_k([0,L])\|^2}} \to -\gamma(k). \tag{20}$$

Thus, the quenched transmission Lyapunov exponent is

$$\mu_q^T(k,p) = \lim_{L\to\infty} \frac{\ln|t(k,L)|^p}{L} = -p\gamma(k) < 0. \tag{21}$$

Calculation of the annealed Lyapunov exponent is more difficult. Typically $r_k \sim e^{L\gamma(k)}$. However, with exponentially small probability $r_k(L)$ can be of the order $e^{-\delta L}, \quad \delta > 0$. Then $\langle r_k^p(L)\rangle = e^{-p\delta L}P\{\ln r_k(L) < -\delta L\}$, and for very negative p the product tends to $+\infty$ (Fig. 1). We use large deviation theory [5] to calculate $\mu_a^T(k,p)$. Let us take $0 \le \beta < \gamma$ and estimate $P\{r_k(L) < e^{\beta L}\}$. Using exponential Chebyshev inequality with optimization over parameter $p \le 0$ we obtain

$$P\{r_k(L) < e^{\beta L}\} = P\{r_k^p(L) > e^{p\beta L}\} \leq \min_{p\leq 0} \frac{\langle r_k^p(L)\rangle}{e^{p\beta L}} \sim \min_{p\leq 0} e^{(\mu_a(k,p)-p\beta)L} = e^{\mu^*(k,p)L}, \tag{22}$$

where $\mu^*(k,\beta) = \max_p(-p\beta + \mu_a(k,p))$ is the Legendre transform [6] of $\mu(k,p)$ for fixed k with respect to parameter p. It is well-known that in the Markov case it is not only estimation from above but the logarithmic equivalence: $P\{r_k < e^{\beta L}\} \overset{\log}{\sim} e^{-\mu^*(k,p)L}$. Now for $p > 0$

$$\langle |t(k,L)|^p\rangle \overset{\log}{\sim} \int \frac{1}{e^{-p\beta L} + e^{p\beta L}}\, dP\{r_k < e^{\beta L}\} = \max_{0\leq\beta\leq\gamma} e^{-p\beta L - \mu^*(k,\beta)} = \begin{cases} e^{\mu(k,-p)L}, & 0 < p \leq 1, \\ e^{\mu(k,-1)L}, & p > 1. \end{cases} \tag{23}$$

For small σ we can use parabolic approximation for $\mu_a^T(k,p)$ that gives

$$\gamma_q^T(k,p) = -p\frac{\pi\widehat{B}(2k)}{4k^2}\sigma^2(1+o(1)) \tag{24}$$

and

$$\mu_a^T(k,p) = \begin{cases} p(p+2)\dfrac{\pi\widehat{B}(2k)}{8k^2}\sigma^2(1+o(1)), & p \leq 1, \\ -\dfrac{\pi\widehat{B}(2k)}{4k^2}\sigma^2(1+o(1)), & p > 1, \end{cases} \tag{25}$$

where $B(x) = \mathrm{Cov}(V(y)V(y+x))$ is the covariance of random potential $V(x)$, and $\widehat{B}(k) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-ikx}B(x)\,dx$ is the corresponding energy spectrum of $V(x)$ (Fig. 1). In particular, for $p = 2$

$$\mu_a^T(k,2) \approx \frac{1}{4}\,\gamma_q^T(k,2) < 0. \tag{26}$$

This relation is the manifestation of the strong intermittency (cf. [1]). It shows that the main contribution to the transmitted energy comes not from "typical" fibers where the logarithmic rate of energy decay is $\gamma_q^T(k,2)$, but rather from few rare fibers (the probability of their occurrence is $e^{\frac{1}{4}\gamma_q^T(k,2)L}$) through which significant part of the energy of order $O(1)$ is transmitted. Thus, we have the I. M. Lifshits picture described in the introduction.

3. Conclusion

We have considered propagation of light through a bundle of independent optical fibers whose refractive index is a random function of length. It is found that distribution of energy at the exit of the bundle has intermittent behavior. For quantitative estimation of irregularity we introduced the generalized energy transmission coefficient and studied its Lyapunov exponent. Essential difference in the quenched and annealed energy transmission Lyapunov exponents is suggested as a manifestation of intermittency. In the case of small randomness of the fiber refractive index it is found that the energy transmission Lyapunov exponent of a typical single fiber is four times bigger than the average one of the bundle. Unlike the moment Lyapunov exponent $\mu_a(p)$ for the amplitude which has quadratic dependence on the moment p, the transmission moment Lyapunov exponent is constant for $p \geq 1$.

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