Regular Coulomb Wave Function Method for Analysis of the Azimuthally Magnetized Circular Ferrite Waveguides

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Abstract— The regular Coulomb wave function method for investigation of the circular waveguides, comprising a co-axially positioned solid ferrite cylinder of azimuthal magnetization that support normal TE_{0n} modes is regarded and put into practice to the simplest case in which the anisotropic medium fills them completely. Its main points are: *i*) the pertinent wave equation is a form of the Coulomb wave equation; *ii*) the field components are expressed by the real Coulomb wave functions $F_L(\eta,\rho)$ where $L=-0.5$ or $L=0.5$, η and ρ — real, $\rho>0$, $-\infty<\eta<+\infty$; *iii*) the solution of propagation problem needs a detailed numerical study of the functions and of their zeros in ρ. The Georgiev and Georgieva's pioneering idea is adopted and advanced to extend with new (in the case considered — with the electromagnetism and in particular with the theory of waveguides) the traditionally established field of application of the Coulomb wave functions $F_L(\eta,\rho)$ and $G_L(\eta,\rho)$ — the atomic physics and quantum mechanics. It stands on the generalized for all complex ρ, η and L Thompson and Barnett's definition of the aforesaid functions, using their relations with the confluent hypergeometric ones. It is proved numerically that the Abramowitz and Stegun's series, expressing the regular function, though initially constructed for real ρ and η ($\rho>0$, $-\infty<\eta<\infty$) and a non-negative integer L ($L=0, 1, 2, \ldots$): *i*) is applicable also for certain fractional positive and negative real L (e.g., $L=\pm0.5$), on condition that the set of allowable values of the other two parameters is unchanged; *ii*) is preferable in the computations, compared to the one, harnessing the Thompson and Barnett's representation of $F_L(\eta,\rho)$ in terms of the complex Kummer function. Its advantage consists in: *i*) it is more rapidly convergent; *ii*) real parameters are used only; *iii*) the independent variable in it is twice smaller. In view of this, employing the expansion of functions around zero, allows practically to get outcomes for much larger values of the latter than the Thompson and Barnett's formula. The mathematical approach described is richly illustrated by graphs.

1. INTRODUCTION

The solution of the problems for normal TE_{0n} modes in the circular waveguides, containing coaxially positioned ferrite cylinder or toroid, magnetized azimuthally with respect to the direction of wave propagation faces serious obstacles of mathematical nature [1–9]. To overcome them different techniques have been proposed, allowing to reveal various aspects of microwave field-anisotropic medium interaction: *i*) Bolle-Heller functions [1], *ii*) transverse network representation [2], *iii*) perturbation techniques [3], *iv*) variational calculus [4], *v*) confluent hypergeometric functions [6–8], *vi*) Coulomb wave functions [5, 9].

The basic features of the regular Coulomb wave function method, employable to configurations in which the ferrite region is cylindrical are considered. The emphasis is placed on: *i*) the original idea by Georgiev and Georgieva to use in the theory of waveguides [5, 9] the generalized for all complex ρ, η and L in Thompson and Barnett's sense [10] regular and irregular (logarithmic) Coulomb wave functions $F_L(\eta,\rho)$ and $G_L(\eta,\rho)$ [11, 12], harnessed until now in the atomic physics and quantum mechanics [11]; *ii*) the possibility to employ the Abramowitz and tegun's representation of the latter [12] for a positive or negative factional real L, appearing in the problem under study, though the functions are defined for non-negative integer values of the parameter mentioned. In view of the geometry chosen, the discussion is confined to $F_L(\eta,\rho)$ solely. The advantages of the new approach in contrast to the one, using the complex Kummer function [10] are pointed out. Graphs are depicted, showing the variation of the generalized function $F_L(\eta,\rho)$ with ρ, provided $L=\pm0.5$ and $\eta=0$, ±1, ±5 and ±10, and of its zeros $o^L_{\eta,n}$ in ρ.

The ferrite is characterized by a Polder permeability tensor $\vec{\mu}=\mu_0[\mu_{ij}]$, i, $j=1$, 2, 3, with non-zero components $\mu_{ii}=1$ and $\mu_{13}=-\mu_{31}=-j\alpha$, $\alpha=\gamma M_r/\omega$, (γ — gyromagnetic ratio, M_r — ferrite remanent magnetization, ω — angular frequency of the wave) and a scalar permittivity $\varepsilon=\varepsilon_0\varepsilon_r$. In the particular case it is thought that it fills-in totally the waveguide. Besides, it is assumed that the latter is infinitely long, lossless and perfectly conducting and that its radius is r_0.

2. CLASSICAL ABRAMOWITZ AND STEGUN'S COULOMB WAVE EQUATION AND REGULAR COULOMB WAVE FUNCTION

The Abramowitz and Stegun's Coulomb wave equation [12]:

$$\frac{d^2\upsilon}{d\rho^2} + \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right]\upsilon = 0, \tag{1}$$

is a second-order ordinary differential equation, determined for ρ and η — real, ($\rho > 0$, $-\infty < \eta < \infty$) and L — a nonnegative integer, ($L = 0, 1, 2, \ldots$). It has regular singularity with indexes $L+1$ and $-L$ at $\rho = 0$ and an irregular one at $\rho = \infty$. Its general integral [12]:

$$\upsilon(\rho) = C_1 F_L(\eta,\rho) + C_2 G_L(\eta,\rho) \tag{2}$$

is written in terms of the regular and irregular (logarithmic) Coulomb wave functions $F_L(\eta,\rho)$ and $G_L(\eta,\rho)$, resp. (C_1 and C_2 are arbitrary constants).

Abramowitz and Stegun give the following standard form of the regular Coulomb wave function [12]:

$$F_L(\eta,\rho) = C_L(\eta)\rho^{L+1}\Phi_L(\eta,\rho) \tag{3}$$

in which

$$C_L(\eta) = \frac{2^L e^{-\pi\eta/2}\left|\Gamma(L+1+j\eta)\right|}{\Gamma(2L+2)} \tag{4}$$

and

$$\Phi_L(\eta,\rho) = \sum_{q=L+1}^{\infty} A_q^L \rho^{q-L-1}. \tag{5}$$

The coefficients A_q^L in series (5) are specified by the expressions:

$$A_{L+1}^L = 1, \tag{6}$$

$$A_{L+2}^L = \frac{\eta}{L+1} \tag{7}$$

and

$$(q+L)(q-L-1)A_q^L = 2\eta A_{q-1}^L - A_{q-2}^L, \quad (q > L+2). \tag{8}$$

3. GENERALIZED REGULAR COULOMB WAVE FUNCTION IN THE THOMPSON AND BARNETT'S SENSE

The Kummer confluent hypergeometric function is determined by the infinite power series [12, 13]:

$$\Phi(a,c;x) = \sum_{p=0}^{\infty} \frac{(a)_p}{(c)_p}\frac{x^p}{p!} \tag{9}$$

which is absolutely convergent for all real or complex values of a, c, x, except $c = l$, $l = 0, -1, -2, \ldots$, $(a)_p = a(a+1)\ldots(a+p-1)$, $(a)_0 = 1$, $(1)_p = p!$, ($p = 0, 1, 2, \ldots$) is the Pochhammer's symbol. $\Phi(a,c;x)$ is an entire analytic function in the whole complex x-plane. It is regular at $x = 0$ and single-valued, wherever it exists. For a fixed x, $\Phi(a,c;x)$ is an entire function of a and a meromorphic one of c with simple poles at the points $c = 0, -1, -2, \ldots$.

In the partial case $a = L+1-j\eta$, $c = 2L+2$ and $x = 2j\rho$ the Kummer function reduces to the regular Coulomb wave one [12]. It holds:

$$\Phi(L+1-j\eta, 2L+2; 2j\rho) = e^{j\rho}F_L(\eta,\rho)\rho^{-L-1}/C_L(\eta) \tag{10}$$

Using this relation Thompson and Barnett proposed a second more universal definition of $F_L(\eta,\rho)$, valid for all complex ρ, η and L by which they made its analytic continuation to each complex plane with cuts along the negative real ρ-axis [10]:

$$F_L(\eta,\rho) = C_L(\eta)\rho^{L+1}e^{-j\rho}\Phi(L+1-j\eta, 2L+2; 2j\rho). \tag{11}$$

4. GEORGIEV AND GEORGIEVA-GROSSE'S APPROACH

As pointed out above, $\Phi(L+1-j\eta, 2L+2; 2j\rho)$ is not defined for $2L+2=l$. Therefore the computation of the regular Coulomb wave function is not possible directly from the Thompson and Barnett's formula (11) for $L=(l-2)/2$ ($L=-1,-3/2,-2,-5/2,\ldots$).

Instead of $\Phi(a,c;x)$ which if c is zero or a negative integer has no sense, F. G. Tricomi suggested to utilize the modified Kummer function [3–5]:

$$\Phi^*(a,c;x)=\frac{1}{\Gamma(c)}\Phi(a,c;x)=\sum_{p=0}^{\infty}\frac{(a)(a+1)\ldots(a+p-1)}{\Gamma(c+p)}\frac{x^p}{p!} \tag{12}$$

that is finite also for $c=1-m$ ($m=1,2,3,\ldots$). Moreover, the important relation [13]:

$$\Phi^*(a,1-m;x)=\lim_{c\to 1-m}\Phi^*(a,c;x)=(a)_m\frac{x^m}{m!}\Phi(a+m,m+1;x) \tag{13}$$

reduces the discussion to a Φ function with c — a positive integer.

In view of (4), applying consecutively (12) and (13), relation (11) could be transformed to give:

$$F_L(\eta,\rho)=2^L e^{-\pi\eta/2}\left|\Gamma(L+1+j\eta)\right|\rho^{L+1}e^{-j\rho}\Phi^*(L+1-j\eta,2L+2;2j\rho) \tag{14}$$

$$F_L(\eta,\rho)=2^{-L-1}e^{-\pi\eta/2}\left|\Gamma(L+1+j\eta)\right|\rho^{-L}e^{-j\rho}(L+1-j\eta)_{-2L-1}\frac{e^{-j\pi(2L+1)/2}}{(-2L-1)!}\Phi(-L-j\eta,-2L;2j\rho) \tag{15}$$

Obviously $-2L=2,3,4,5,\ldots$; $-2L-1=1,2,3,4,\ldots$; $-L=1,3/2,2,5/2,\ldots$ $-L-1=0,-1/2,-1,-3/2,\ldots$.

Accordingly, Georgiev and Georgieva-Grosse suggested to use formula (11) for finding the numerical values of $F_L(\eta,\rho)$, valid for all complex ρ, η and L, except $L=(l-2)/2$ and (15) provided the letter holds. These authors have made a numerical experiment. The evaluation of real regular Coulomb wave function in case of fractional positive and negative L (e.g., $L=-0.5$ and $L=0.5$) has been performed, following two approaches:

i) direct computation by means of the Abramowitz and Stegun's representation (3)–(8),

ii) calculation through the Thompson and Barnett's expansion of $F_L(\eta,\rho)$ in terms of the complex Kummer confluent hypergeometric function.

The juxtaposition of results obtained has shown perfect coincidence which is a criterion for their reliability. Hence, it could be concluded that though Abramowitz and Stegun's series $\Phi_L(\eta,\rho)$ is defined only for L — a nonnegative integer, it could be used also for fractional values of L. Moreover, the original series for $F_L(\eta,\rho)$ (3)–(8) is much more rapidly convergent in the case mentioned and is preferable in the calculations than (11). Besides, it could provide results for larger values of parameter ρ. The reason for this is the fact that the series $\Phi_L(\eta,\rho)$ which in fact is computed is real whereas the main element of (11) $\Phi(L+1-j\eta,2L+2;2j\rho)$ is complex. Further, for fixed ρ the value of $F_L(\eta,\rho)$ from (11) is found out for the twice larger purely imaginary variable $2j\rho$.

5. REGULAR COULOMB WAVE FUNCTION METHOD

The solution of Maxwell equations in the anisotropic medium shows that the longitudinal component of the rotationally symmetric $TE(H_r,E_\theta,H_z)$ modes satisfies the following second-order ordinary differential equation [9]:

$$\left(r^{-1}D_r r D_r+\beta_f^2-\beta^2-\alpha\beta\, r^{-1}\right)H_z=0 \tag{16}$$

in which $D_r=\partial/\partial r$ is a differential operator, $\beta_f^2=\beta_0^2\varepsilon_r\mu_{eff}$, $\beta_0^2=\omega^2\varepsilon_0\mu_0$ and β are the phase constants in the unlimited space, occupied by azimuthally magnetized ferrite, and the ones in the free space and in the geometry examined, resp., and $\mu_{eff}=1-\alpha^2$ is the effective relative permeability of the load. If $\beta_f>\beta$ ($\hat{\beta}_f<\hat{\beta}$) it governs the propagation of normal TE_{0n} modes along the structure. Putting $\rho=\beta_2 r$, $\beta_2=(\beta_f^2-\beta^2)^{1/2}$ — radial wavenumber of the normal TE_{0n} modes in the anisotropic medium, $H_z(r)=\upsilon(\rho)\rho^{-1/2}$, $\eta=\alpha\beta/(2\beta_2)$ and $L=-0.5$ (ρ, β_2, η, L — real, $\rho>0$, $\beta_2>0$, $L<0$) reduces Equation (16) to the standard M. Abramowitz form of the Coulomb wave Equation (1).

Accordingly, the field components and the characteristic equation of the normal modes are given by the expressions:

$$H_r = jB\left[\frac{\left(1-\alpha^2\right)^{1/2}\beta}{\Gamma_f}\frac{F_{L+1}(\eta,\rho)}{\rho^{L+1}} + \alpha\frac{F_L(\eta,\rho)}{\rho^{L+1}}\right], \tag{17}$$

$$E_\theta = -j\frac{B\omega\mu_0\left(1-\alpha^2\right)^{1/2}}{\Gamma_f}\frac{F_{L+1}(\eta,\rho)}{\rho^{L+1}}, \tag{18}$$

$$H_z = B\frac{F_L(\eta,\rho)}{\rho^{L+1}}, \tag{19}$$

$$F_{L+1}(\eta,\rho_0) = 0 \tag{20}$$

where B is an arbitrary constant, $L = -0.5$ and $\rho_0 = \beta_2 r_0$. As seen, the Equations (17)–(20) depend on the ferrite remanent magnetization (on α, resp. η). Correspondingly, it might be expected the change of the magnitude and sign of the latter will impact the field distribution and phase diagram of the $TE(H_r, E_\theta, H_z)$ waves, described by it.

It is instructive to introduce the normalized (barred) quantities: $\bar{\beta} = \beta/(\beta_0\sqrt{\varepsilon_r})$, $\bar{\beta}_2 = \beta_2/(\beta_0\sqrt{\varepsilon_r})$, $\bar{r}_0 = \beta_0 r_0\sqrt{\varepsilon_r}$ and $\eta = \alpha\bar{\beta}/(2\bar{\beta}_2)$, and to rewrite Equations (17)–(20) in terms of them. Then, if $o^L_{\eta,\,n}$ stands for the nth ($n = 1, 2, 3, \ldots$) consecutive zero of $F_L(\eta,\rho)$ in ρ for given η, Equation (20) holds when $\bar{\beta}_2 = o^{L+1}_{\eta_\pm,n}/\bar{r}_0$ which specifies the eigenvalue spectrum of the transmission line.

6. NUMERICAL STUDY OF THE ZEROS OF THE REGULAR COULOMB WAVE FUNCTION

To plot the phase characteristics of the waveguide and to find the field distribution in it, requires a numerical investigation of the Coulomb function and its zeros in the real area, corresponding to the normal TE_{0n} modes for the values of parameters, given in the previous Section. In the computations the Abramowitz and Stegun's series (3)–(8) is used. The results are confronted with the ones, obtained from the Thompson and Barnett's formula (11), showing a perfect coincidence for small ρ, η and L. Figures 1 and 2 (Figures 3 and 4) visualize the change of $F_L(\eta,\rho)$ with ρ in the interval $\rho \in [0, 25]$ in case $L = -0.5$ ($L = 0.5$), assuming η as parameter. The solid (dashed) lines in the first (second) of them concur to $\eta > 0$, $\eta = 1$, 5 and 10 ($\eta < 0$, $\eta = -1$, -5 and -10). The dotted curve answers to $\eta = 0$. As seen, $F_L(\eta,\rho)$ is oscillating. When ρ gets large, it behaves like a sine function which might be expected in view of its asymptotic representation [12]. The value of independent variable at which this is exhibited, depends on the other two parameters. At

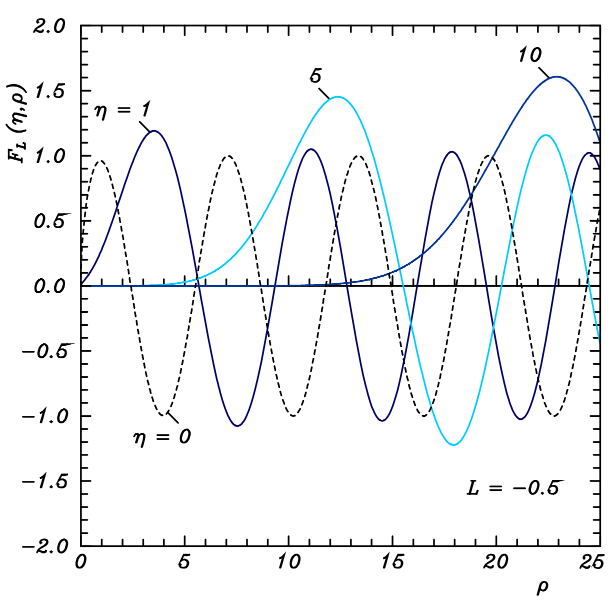

Figure 1: Regular Coulomb wave function $F_L(\eta,\rho)$ vs. ρ for $L = -0.5$ and $\eta = 0$, 1, 5, 10.

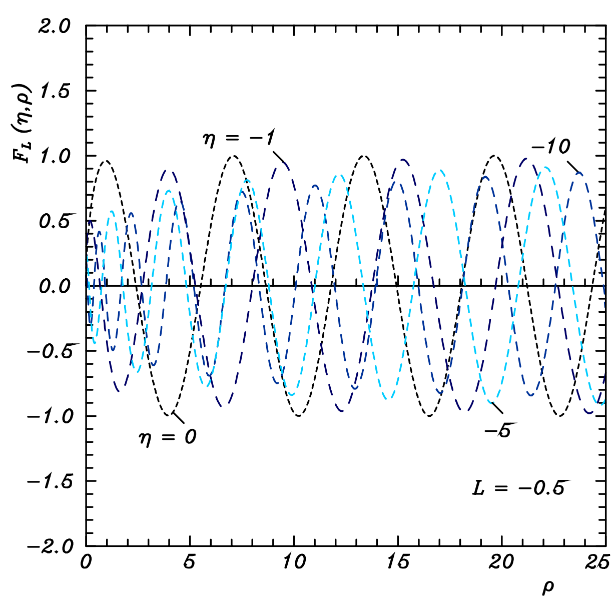

Figure 2: Regular Coulomb wave function $F_L(\eta,\rho)$ vs. ρ for $L = -0.5$ and $\eta = 0$, -1, -5 and -10.

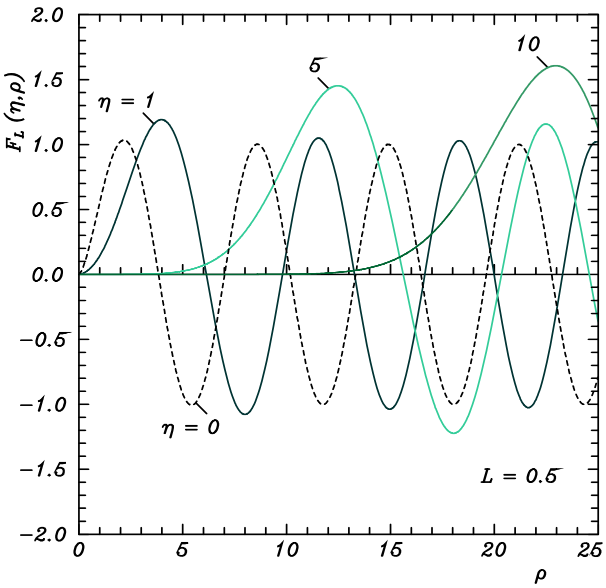

Figure 3: Regular Coulomb wave function $F_L(\eta,\rho)$ vs. ρ for $L = 0.5$ and $\eta = 0$, 1, 5, 10.

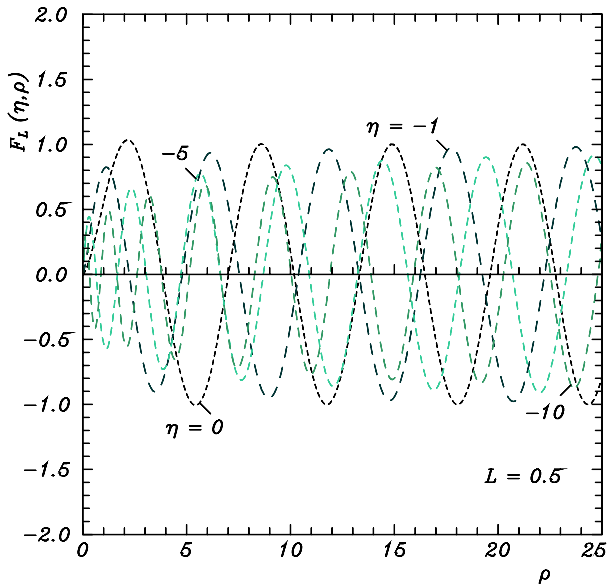

Figure 4: Regular Coulomb wave function $F_L\ (\eta,\rho)$ vs. ρ for $L = 0.5$ and $\eta = 0$, -1, -5 and -10.

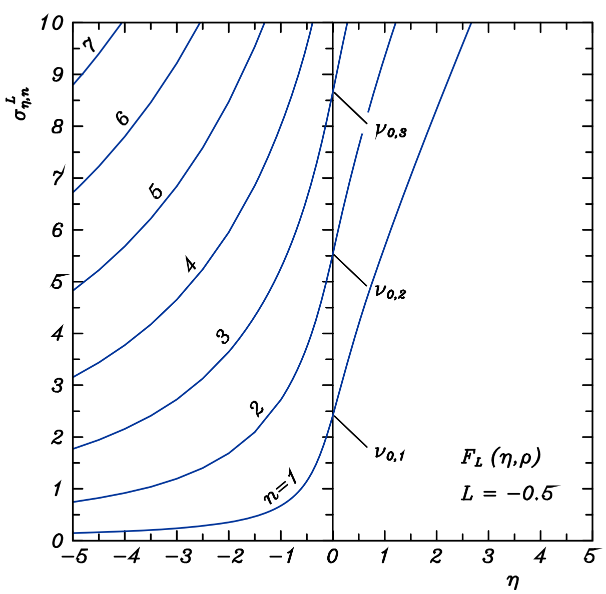

Figure 5: Zeros $o^L_{\eta,n}$ of $F_L(\eta,\rho)$ in case $L = -0.5$ as a function of η for $n = 1$ (1) 4.

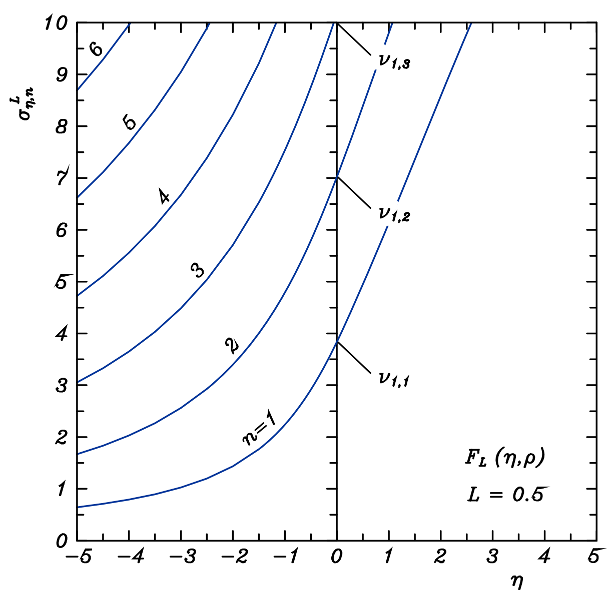

Figure 6: Zeros $o^L_{\eta,n}$ of $F_L(\eta,\rho)$ in case $L = 0.5$ as a function of η for $n = 1$ (1) 4.

given L the quasi-sine oscillations are displayed for smaller ρ, provided η diminishes. For identical η this tendency is more pronounced for larger ρ, if L increases. In addition, the maxima of the first oscillations of $F_L(\eta,\rho)$ grow with η and L. On this condition all maxima are shifted to larger ρ.

The regular Coulomb function for fractional positive and negative L has an infinite number of zeros $o^L_{\eta,n}$ in ρ, like that for a non-negative integer L. The dependence of the first several of them for $\eta \in [-5, 5]$ is portrayed in Figures 5, 6, provided $L = -0.5$ and $L = 0.5$. The symbols $\nu_{0,n}$ and $\nu_{1,n}$ in them stand for the nth zeros of the zeroth and first order Bessel functions $J_0(\rho)$ and $J_1(\rho)$, resp., (see the second Kummer theorem [12]). As seen, for fixed L and n (η), $o^L_{\eta,n}$ monotonously increase with η (grow with both L and n). Further, the numerical analysis shows that it holds:

$$\lim_{\eta_- \to -\infty} o^L_{\eta_-,n} = 0, \tag{21}$$

$$\lim_{\eta_+ \to +\infty} o^L_{\eta_+,n} = +\infty, \tag{22}$$

$$\lim_{\eta_- \to -\infty} |\eta_-| \, o^L_{\eta_-,n} = T_1 (L, n) , \tag{23}$$

where $T_1(L, n)$ are finite positive real numbers. Assuming $L = -0.5$ and $n = 1$, (1), 5, it is fulfilled: $T_1(L, n) = 0.72289\ 82454$, 3.80890 77930, 9.36087 58488, 17.38003 55533, 27.86653 79522, resp. In case $L = 0.5$ for the same values of n, it is true: $T_1(L, n) = 3.29682\ 70534$, 8.85624 98649, 16.87758 86082, 27.36502 36432, 40.31938 95366, resp.

7. PHYSICAL INTERPRETATION OF THE OUTCOMES FOR THE ZEROS

The analysis of the results presented for the zeros of $F_L(\eta, \rho)$, combined with that of the functions and with the peculiarities of the problem for the normal TE_{0n} modes examined, discloses the most important features of the phase behaviour of the circular waveguide, entirely filled with azimuthally magnetized ferrite: *i*) The number of modes which might get excited in the configuration is infinite, equal to the number of roots, resp. zeros $o^{L+1}_{\eta,n}$ of the concurring characteristic Equation (20), resp. wave function n in case $L+1 = 0.5$. *ii*) The waves might be sustained both for positive and negative magnetization of the anisotropic load; *iii*) The area of propagation of normal waves, observed for $-1 < \alpha < 1$, assuming negative (positive) magnetization, is bilaterally restricted (is limited from below and unbounded from the side of higher frequencies); *iv*) The outcomes for the zeros both at $L = -0.5$ and $L + 1 = 0.5$ are useful in the analysis of the field distribution.

8. CONCLUSIONS

The approach for examination of the circular waveguides, containing a co-axial azimuthally magnetized ferrite cylinder which support normal TE_{0n} modes, based on the real regular Coulomb wave functions of specially chosen parameters, is regarded. The discussion is focused on the structure, completely filled with the anisotropic medium mentioned. An analysis is accomplished of the methods for computation of the functional values and of certain of their zeros, necessary for the integration of the task formulated. The results are presented in graphical form and discussed. The main characteristics of wave propagation are discussed.

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