Mutual Inductance of Two Helical Coils — Theory, Calculation, Verification

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Abstract— The paper deals about the verification of calculation process of a mutual inductance between two helical coils, whereby backward computation of voltages and currents of designed wireless energy transfer system is serving for confirmation of proposed methodology.

Nowadays, there are many mathematical procedures how to compute mutual inductance between two coils. The main approach of proposed paper is exact specification of the formula, which may be used for the determination of the mutual inductance. After mathematical computation were done the confirmation of validity of proposed formula for mutual inductance calculation is verified with simulation as well as with experimental measurement on proposed test bench of wireless energy system. In order to determine the accuracy of investigated computation procedure backward confirmation has been provided in the way of comparison of time-waveforms of input/output variables (voltages/currents) from measurement experiment on proposed wireless power transfer (WPT) system with calculated time-waveforms.

1. INTRODUCTION — GENERAL THEORY

One of the important parameter for wireless power transmission is the mutual inductance that tells us about how much is the electromagnetic coupling between two cylindrical coils strong, when an air is located between them, and when they are positioned without coaxial and angular deflection.

Mutual inductance between two coils originates from the magnetic flux, which is generated by one of the coils. For the sake of simplicity, let imagine two coils with one turn, whereby such a coil can be defined as a loop. Transmitting loop generates magnetic flux Φ_1 and electromagnetic force ε. These variables are result of time-varying current flowing through transmitter. A part of generated flux Φ_1 is enveloped by receiving coil. This flux is designated as Φ_{12} and is defined as mutual flux. Mutual flux for given area of coil generates magnetic induction, which is given by Biot-Savavarth-Laplace law [X]-[X].

Mutual magnetic flux Φ_{12}, which flows through the area of receiving coil S can be defined as follows:

$$\Phi_{12} = \int_{S_z} B_2 dS_1 = \left(\frac{\mu_0}{4\pi}\int_{S_1}\oint_{l_2}\frac{dl_2 \times r_{21}}{r_{21}^3}dS_1\right) I_2 \tag{1}$$

In the same way the magnetic flux Φ_{21} can be defined and as valid for the reverse flow (i.e., receiving coil act as source).

$$\Phi_{21} = \int_{S_1} B_1 dS_2 = \left(\frac{\mu_0}{4\pi}\int_{S_2}\oint_{l_1}\frac{dl_1 \times r_{12}}{r_{12}^3}dS_2\right) I_1 \tag{2}$$

Formulas (1) and (2) can be further simplified into formulas (3) and (4):

$$\Phi_{12} = L_{12} I_2 \tag{3}$$
$$\Phi_{21} = L_{21} I_1 \tag{4}$$

Consequently for the calculation of mutual inductances, next formulas are valid:

$$L_{12} = \frac{\mu_0}{4\pi}\int_{S_1}\oint_{l_2}\frac{dl_2 \times r_{21}}{r_{21}^3}dS_1 \tag{5}$$
$$L_{21} = \frac{\mu_0}{4\pi}\int_{S_2}\oint_{l_1}\frac{dl_1 \times r_{12}}{r_{12}^3}dS_2 \tag{6}$$

Inductances L_{12} and L_{21} are the same. This fact can be confirmed, when we express magnetic induction B in the way of vector potential [X]. Based on this, it can be stated that $L_{12} = L_{21} = M$, whereby M is defined as mutual inductance, whose static formula is as follows:

$$M = \frac{\Phi_{21}}{I_1} = \frac{\Phi_{12}}{I_2} \tag{7}$$

The meaning of several symbols and variables is interpreted on Fig. 1.

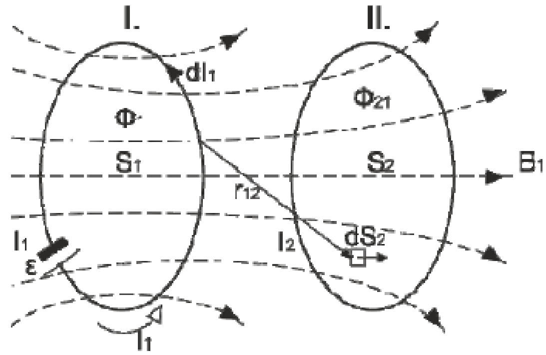


Figure 1: Mutual magnetic coupling of two closed loop.

2. COMPUTATION FORMULAS FOR DETERMINATION OF MUTUAL INDUCTANCE

For calculation of the magnetic flux for round coil use of vector's potential A is better choice, instead of magnetic induction B. Vector's potential is defined by

$$B = \operatorname{rot} A \tag{8}$$

Then magnetic flux can be defined by integration of the vector's potential along circumference of the current turn.

$$\Phi = \oint_l A \cdot dl \tag{9}$$

Advantage of using vector's potential is that, we can work with it's as a scalar quantity, for a coaxial arrangement of the round coils. Vector's potential driven by thin current turn can be written as

$$A_\varphi(r, \varphi, z) = \frac{\mu_0 I}{4\pi} R_1 \int_0^{2\pi} \frac{\cos\varphi'}{\sqrt{r^2 + R_1^2 - 2rR_1\cos\varphi' + (z - z')^2}} d\varphi' \tag{10}$$

For better explanation of (10), see [1]. Magnetic flux for two coaxial thin round coils can by written as:

$$\Phi_{C/C} = \frac{\mu_0}{2} \frac{N_1}{h_1} \frac{N_2}{h_2} R_1 R_2 I \cdot \int\limits_{z=z_{21}}^{z=z_{22}} \left(\int\limits_{z'=z_{11}}^{z'=z_{12}} \left(\int_0^{2\pi} \frac{\cos\varphi'}{\sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos\varphi' + (z - z')^2}} d\varphi' \right) dz' \right) dz \tag{11}$$

Mutual inductance for two coaxial, thin, round coil can be defined by substituting (11) to the (7)

$$M_{C/C} = \frac{\mu_0}{2} \frac{N_1}{h_1} \frac{N_2}{h_2} R_1 R_2 \cdot \int\limits_{z=z_{21}}^{z=z_{22}} \left(\int\limits_{z'=z_{11}}^{z'=z_{12}} \left(\int_0^{2\pi} \frac{\cos\varphi'}{\sqrt{R_1^2 + R_2^2 - 2R_1R_2\cos\varphi' + (z - z')^2}} d\varphi' \right) dz' \right) dz \tag{12}$$

This expression can be rewritten by elliptical integrals as:

$$M_{C/C} = \frac{4}{3}\mu_0 (R_1 R_2)^{\frac{3}{2}} \frac{N_1}{h_1} \frac{N_2}{h_2} \cdot [X(k_{11}) - X(k_{22}) - X(k_{33}) + X(k_{44})] \tag{13}$$

where function $X(k)$ is defined as:

$$X(k) = \frac{1}{k} \left[\frac{1 - k^2}{k^2} (K(k) - E(k)) + \frac{3\rho - 4}{2} E(k) - \frac{3}{2} \rho (1 - k^2) \Pi \left(\frac{\rho k^2 - 2}{\rho - 2}, k \right) \right] \tag{14}$$

where $K(k)$ is elliptical integral of first kind, $E(k)$ is elliptical integral of second kind and $\Pi(n, k)$ is elliptical integral of third kind. Elliptical integrals and relevant modules won't be described in detail here, but the relevant information can be found in [2].

3. CALCULATION AND EXPERIMENTAL MEASUREMENT OF MUTUAL INDUCTANCE

The computation algorithm for calculation of self-inductance and mutual inductance of two air coils is written in Matlab/Simulink. This can be divided into three parts (Fig. 2). First part is part, where we define main parameters of both coils, e.g., radius or height of coil. Main parameters of coils were taken over from existing system for wireless energy transfer (Fig. 3). Second part is calculation of self-inductance. At first, the program calculates module the elliptic integrals and then next subsystem calculates self-inductance. Then in next subsystem mutual inductance by using (13) is being calculated. The results from computation are listed in Table 2.

As was already mentioned, the main parameters of the coils were taken over from existing system for wireless power transfer (Fig. 3), which is primarily designed for the purposes of mutual inductance investigation. The transmitting and receiving coils were designed with helical geometry. As high frequency generator the evaluation board EPC9003 has been used [6]. Its main specification is the use of perspective GaN power transistor devices (eGaN EPC2010C) which are suitable for very high frequency operation.

Table 1 shows characteristic parameters of proposed helical coils for wireless energy transfer. These parameters have been computed based on the target application of wireless system, whose main parameters are parasitic resistance R and quality factor Q. Quality factor is derived from other coil's parameter. Target application of future proposal of this system shall be wireless charging of

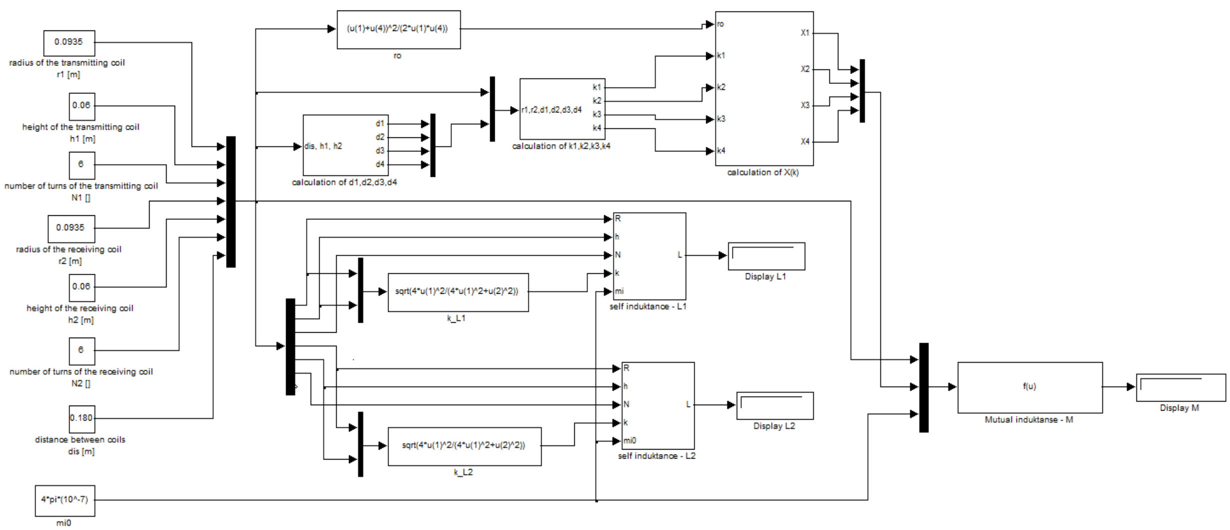
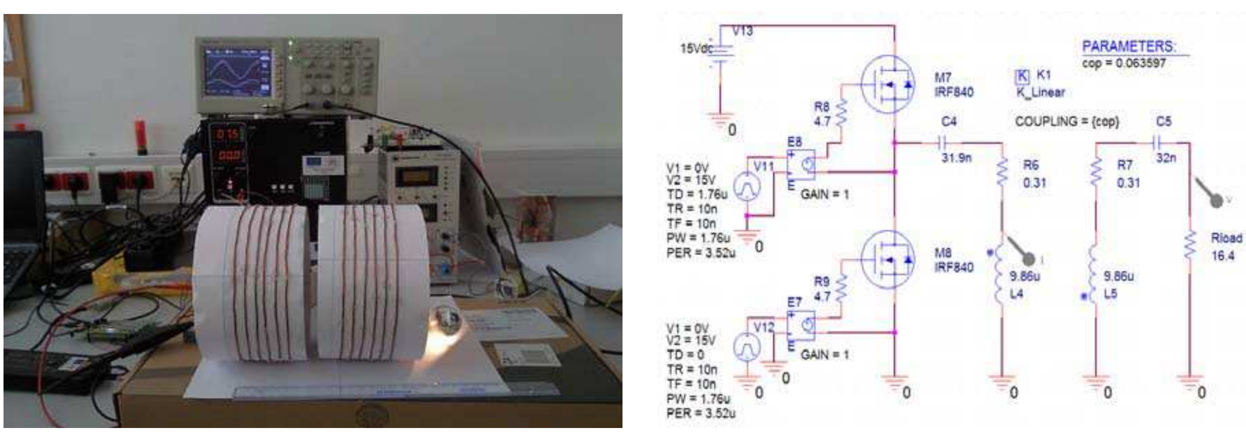


Figure 2: GUI of computation algorithm for mutual and self-inductance calculation of two helical coils.



(a) (b)

Figure 3: (a) Experimental test bench of wireless power transfer, for the purposes of mutual inductance investigation and (b) principal schematics of transmitter and receiver configuration.

e-vehicles, thus existing model serves as testing sample in reduced ratio. After coil construction, the main parameters influencing power transfer have been measured (self-inductance, parasitic resistance). These parameters are listed in next table.

For the investigation of mutual inductance between transmitter and reciever, it was necessary to measure input current of transmitting coil I_{Lp} and voltage on the load U_Z. This is due to fact, that mutual inductance can not be measured directly. The formula for the computation is:

$$M = \frac{U_z}{\omega I_{Lp}} \tag{15}$$

Consequently we have determined the relative error between measured and computed values of mutual inductance.

$$\Delta[\%] = \frac{M_{mes} - M_{cal}}{M_{cal}} \times 100 \tag{16}$$

4. BACKWARD CONFIRMATION OF MATHEMATICAL CALCULATION OF MUTUAL INDUCTANCE

Due to future purposes, where we would like to continue in the design of WPT for electro-mobility applications, the simulation designing with mathematical approach of mutual inductance determination acts as flexible solution. The advantages of sim-math prototyping for the target application are fast achievement of results, more flexible system optimization and no need of physical system disposal. Due to these facts, the reliable simulation models must be prepared. Here we would like to confirm the mathematical determination of mutual inductance in the way of mathematical computation of time waveforms of input and output variables (voltages/currents) of proposed WPT system. The results from mathematical computation are consequently compared with simulation results from OrCAD PSpice, as well as with experimental measurements on physical sample.

Table 1: Computed parameters of proposed coils for wireless power transfer.

Param.	**Description**	**Value**	**Unit**
D	Diameter of a air coil	185	mm
l	Coil length	60	mm
a	Wire diameter	1.5	mm
N	Number of turns	6	
p	Pitch between turns	10	mm
l_w	Physical length of used wire	3486	mm
L	Inductance	9.34	uH
R	Effective serial AC resistance	0.062	Ohm
C	Parasitic capacitance	1020	pF
Q	Quality factor	174	
Lp	Inductance	9.86	uH
Ls	Inductance	10.12	uH
Rp	Serial DC resistance	1.09	Ohm
Rs	Serial DC resistance	0.8	Ohm

Table 2: Measured parameters on a WPT system for mutual inductance calculation.

dis [mm]	$\mathbf{U_Z}$ **[V]**	$\mathbf{I_{Lp}}$ **[A]**	$\mathbf{M_{mea}}$ **[μH]**	$\mathbf{M_{cal}}$ **[μH]**	**Δ [%]**
100	26.6	8.4	1.75362	1.6247	6.54
120	23.8	10.4	1.2673	1.1949	4.56
140	19.8	11.6	0.94523	0.899	3.59
160	15.8	12.4	0.7056	0.6891	7.88
180	13.2	12.8	0.571	0.537	4.59

The computation of input/output variables of proposed WPT system was done with the use of next formulas (17–20). Due to complexity and mutual dependency of these variables, the GUI in Matlab has been developed (Fig. 4).

$$\frac{du_{C_1}}{dt} = \frac{1}{C_1} i_{L_1} \tag{17}$$

$$\frac{du_{C_2}}{dt} = \frac{1}{C_2} i_{L_2} \tag{18}$$

$$\begin{aligned}\frac{di_{L_1}}{dt} &= -\frac{L_2}{L_1L_2 - M^2} u_{C_1} + \frac{M}{L_1L_2 - M^2} u_{C_2} - \frac{L_2\left(R_1\right)}{L_1L_2 - M^2} i_{L_1} \\ &\quad + \frac{M\left(R_2 + R_{load}\right)}{L_1L_2 - M^2} i_{L_2} + \frac{L_2}{L_1L_2 - M^2} u_{IN}\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{di_{L_2}}{dt} &= \frac{M}{L_1L_2 - M^2} u_{C_1} - \frac{L_1}{L_1L_2 - M^2} u_{C_2} + \frac{M\left(R_1\right)}{L_1L_2 - M^2} i_{L_1} \\ &\quad - \frac{L_1\left(R_2 + R_{load}\right)}{L_1L_2 - M^2} i_{L_2} - \frac{M}{L_1L_2 - M^2} u_{IN}\end{aligned} \tag{20}$$

Next figures (Fig. 5–Fig. 8) show time waveforms of the input/output variables which were displayed based on the results from computation (17)–(20). The main parameter, which was changing, is the distance between transmitting and receiving coil (100 mm, 160 mm), thus in principle the value of mutual inductance has been changed. To confirm the accuracy of the data, we used the comparison with the experimental measurements, where distance between coils was also changed similarly like in the case of computation. Here we would like to note that for the time waveforms from measurements (Fig. 6, Fig. 8), the phase shift between voltage and current is visible. This phase shift is caused due to insufficiency of measuring equipment, more exactly due to low measuring frequency range of current probe (HAMEG HZ 56) which is up to 100 kHz (the switching frequency of proposed WPT system is 300 kHz).

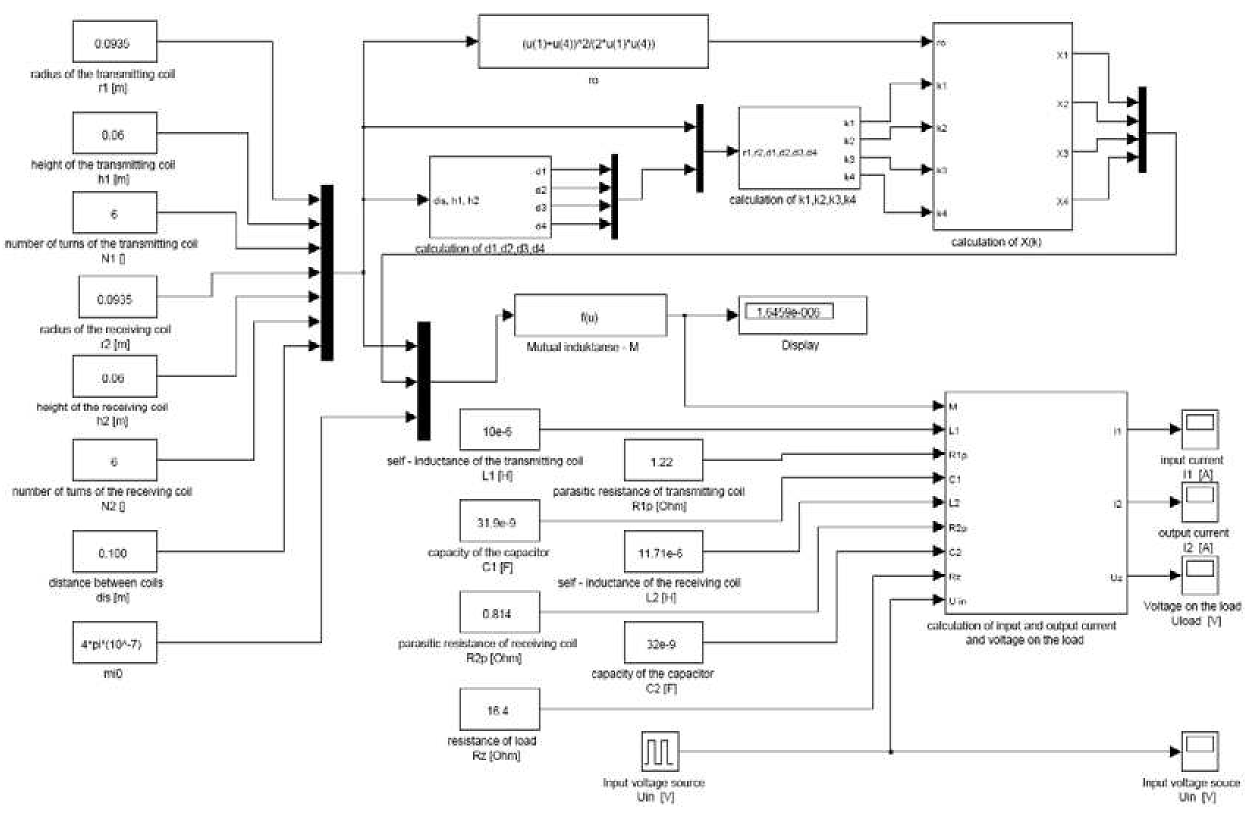
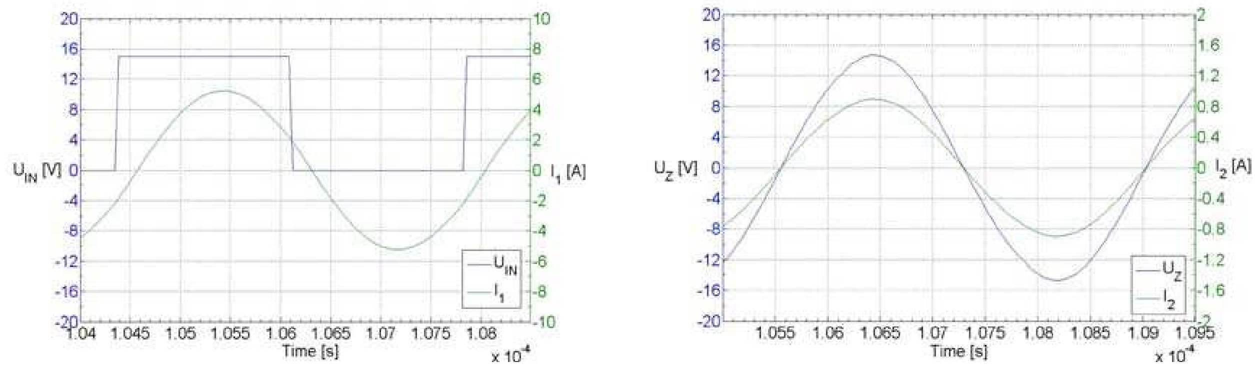
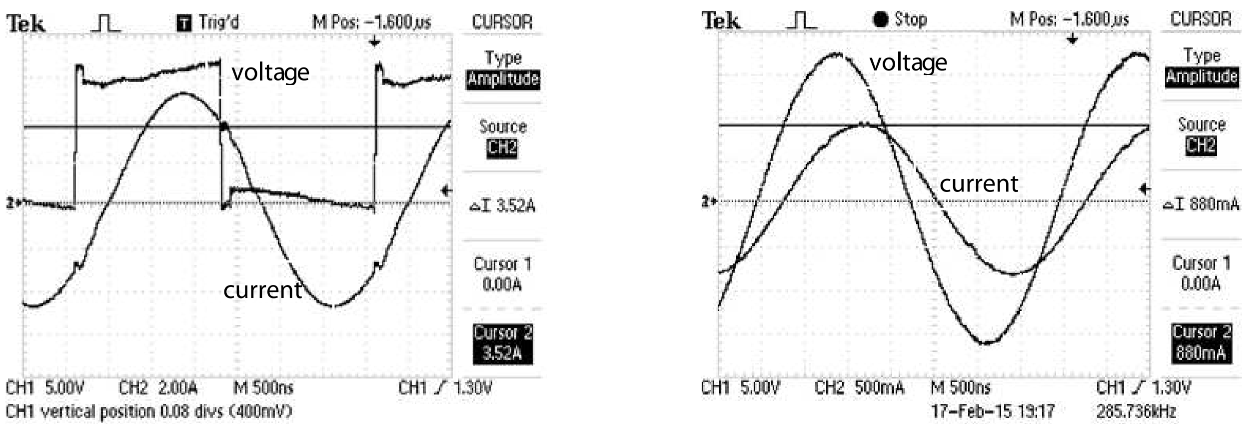


Figure 4: Matlab GUI for computation of input/output variables of WPT system.



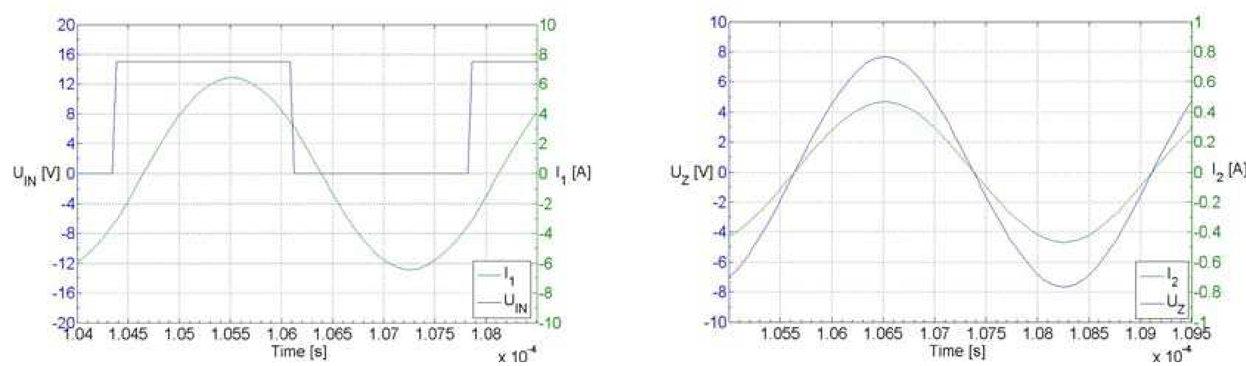
(a) (b)

Figure 5: (a) Voltage and current on transmitting coil and voltage and (b) current on receiving coil for the distance of 100 mm (results from computation).



(a) (b)

Figure 6: (a) Voltage and current on transmitting coil and (b) voltage and current on receiving coil for the distance of 100 mm (results from experimental measurement).



(a) (b)

Figure 7: (a) Voltage and current on transmitting coil and (b) voltage and current on receiving coil for the distance of 160 mm (results from computation).

In next table, the comparisons of the magnitudes of previously investigated variables are listed together with the value of relative error.

Based on that, we can say that the proposed determination of the mutual inductance and also proposed computational algorithm of time-waveforms of variables for WPT system can be considered as reliable tool for future purposes, which will be related with design of the concept of WPT for electromobility applications. The designed procedure is very attractive from the speed

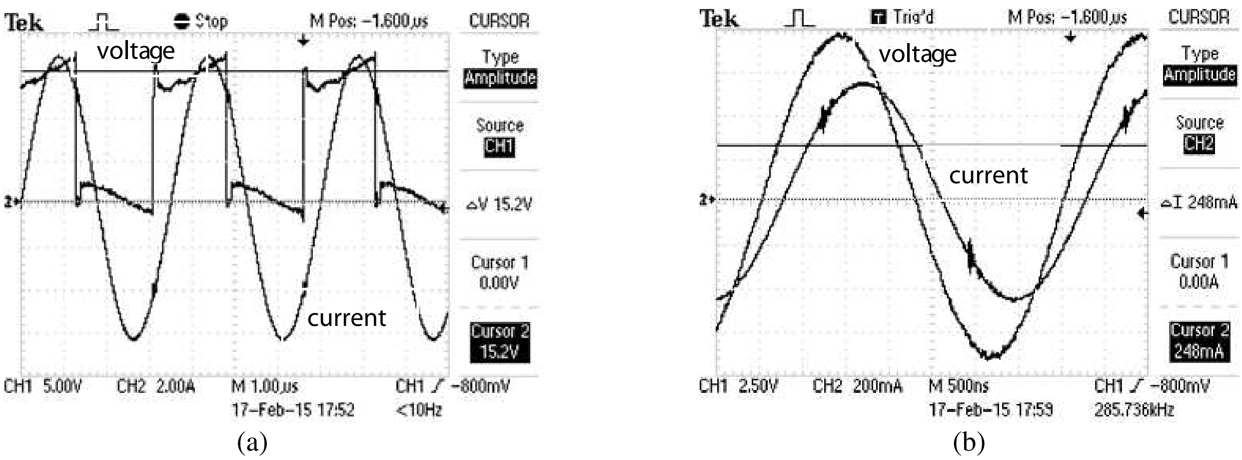


Figure 8: (a) Voltage and current on transmitting coil and (b) voltage and current on receiving coil for the distance of 160 mm (results from experimental measurement).

Table 3: Comparisons of magnitude input/output variables of proposed WPT system.

	computation				measurement			
	$\mathbf{U_{1\,m}}$ [V]	$\mathbf{I_{1\,m}}$ [A]	$\mathbf{U_{2\,m}}$ [V]	$\mathbf{I_{2\,m}}$ [A]	$\mathbf{U_{1\,m}}$ [V]	$\mathbf{I_{1\,m}}$ [A]	$\mathbf{U_{2\,m}}$ [V]	$\mathbf{I_{2\,m}}$ [A]
$\boldsymbol{d = 100}$ **[mm]**	15	5.2	14.8	0.9	15	5.25	17.2	0.95
$\boldsymbol{d = 160}$ **[mm]**	15	6.3	7.5	0.47	15	6.9	8.7	0.55

Relative error [%] ($\boldsymbol{d = 100}$ mm)	15	5.2	14.8	0.9	15	5.25	17.2	0.95
Relative error [%] ($\boldsymbol{d = 160}$ mm)	15	6.3	7.5	0.47	15	6.9	8.7	0.55

of design point of view. A lot of physical designing can be eliminated, thus optimization of WPT system can be done quickly with the used of proposed procedure.

The only problem which must be solved in order to improve accuracy of the results, is exact investigation and consequent definition of parasites, which are frequency dependent, and are influencing efficiency of WPT (parasitic resistances, capacitances of transmitter and receiver).

5. CONCLUSION

In this paper, we were forced to evaluate the accuracy of the computation of mutual inductance between two coils, whose might be suited for wireless energy transfer. The algorithm for the computation of mutual inductance was designed based on [1].

The received results show that measured values are in acceptable accordance with computed results, whereby maximal relative error was 7.88%. The main process which probably influenced the accuracy of measurement was existence of frequency and temperature dependent parasitic components, mainly ESR of capacitors (it was series connection of set of capacitors). Also backward confirmation of mutual inductance determination was realized in the way of comparisons of the input/output variables of proposed WPT system. Computed and measured values were compared, whereby the highest relative error was 17.2%. The main parameter, which influences these differences, is temperature and frequency dependency of parasitic elements. These facts must be included into considerations in order to improve received results.

Future work will be focused on investigation of parasitic components, which are influencing mutual inductance. Optimization process for their suppression will be done.

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