Theorem for the $G_1(c, n)$ Numbers

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Abstract— The theorem for existence of the $G_1(c, n)$ numbers (finite real positive numbers, coherent with the zeros $\kappa_{z,n}^{(c)}$ in the imaginary part κ of the complex first parameter a of the complex Kummer confluent hypergeometric function $\Phi(a, c; x)$ with $a = c/2 + j\kappa$ — complex, $c = 2\mathrm{Re}a$ — restricted positive integer ($c = 1, 2, 3, \ldots$), κ — real (positive, negative or zero, $-\infty < \kappa < +\infty$), $x = jz$ — positive purely imaginary, z — real, positive and $n = 1, 2, 3, \ldots$) is formulated and proved numerically. It serves as a definition of quantities in question and determines them as the limit of the infinite sequence of real numbers $\{D_1(c, n, z)\}$ ($D_1(c, n, z) = \kappa_{z,n}^{(c)}\, z$) for $z \to 0$. Supposedly this assumption holds, the theorem states that the sequence of the zeros $\{\kappa_{z,n}^{(c)}\}$ is divergent and its terms become infinitely large positive. Moreover, if $z \to +\infty$ both $\{\kappa_{z,n}^{(c)}\}$ and $\{D_1(c, n, z)\}$ are divergent and tend to $-\infty$. Tables and graphs illustrate the influence of parameters c and n on $G_1(c, n)$. The benefit of numbers is manifested in the theory of azimuthally magnetized circular ferrite waveguides, propagating normal TE_{0n} modes.

1. INTRODUCTION

The zeros of Kummer confluent hypergeometric function $\Phi(a, c; x)$ [1] with $a = c/2 - jk$ — complex, c — real, ($c \neq l$, $l = 0, -1, -2, -3, \ldots$), k — real, $x = jz$, z — real, positive and $n = 1, 2, 3, \ldots$ are of special interest in the theory of circular anisotropic waveguides of azimuthal magnetization [2–6]. Until now the attention has been the focused mainly on the positive purely imaginary zeros $\zeta_{k,n}^{(c)}$ of $\Phi(a, c; x)$ with respect to its independent variable z [2–6]. It has been established that for them it holds: $\lim_{k_+ \to +\infty} \zeta_{k_+,n}^{(c)} = +\infty$ and $\lim_{k_- \to -\infty} \zeta_{k_-,n}^{(c)} = 0$, where the subscripts "+" ("−") label quantities, corresponding to positive (negative) sign of k ($k_+ > 0$ and $k_- < 0$) [2]. The numerical analysis has shown that if $c \neq l$ the sequences $\{K_-(c, n, k_-)\}$ and $\{M_-(c, n, k_-)\}$ ($K_-(c, n, k_-) = |k_-|\zeta_{k_-,n}^{(c)}$ and $M_-(c, n, k_-) = |a_-|\zeta_{k_-,n}^{(c)}$) are convergent provided $k_- \to -\infty$, though the multiples in the products $|k_-|\zeta_{k_-,n}^{(c)}$ and $|a_-|\zeta_{k_-,n}^{(c)}$ become unrestricted, resp. infinitesimal. They tend to finite real positive limits, called $L_1(c, n)$ numbers (denoted by $L(c, n)$, too) [2, 4, 5, 7–9]. The definition of the latter has been extended for $c = l$, as well [7–9]. Later on a similar property of the positive purely imaginary zeros in z of certain special functions, incorporating several complex confluent and possibly also a few real cylindrical ones of specially selected parameters has been found out [8–11]. The discussion has been extended to the real case, too [11].

Recently, an investigation of the zeros $\kappa_{z,n}^{(c)}$ in κ of the complex Kummer function $\Phi(a, c; x)$ of $a = c/2 + j\kappa$ — complex, $c = 2\mathrm{Re}a$ — restricted positive integer ($c = 1$, 2 and 3), κ — real, $x = jz$ — positive purely imaginary, z — real, positive and $n = 1, 2, 3, \ldots$ has been made, assuming z as parameter. It has been established that when z diminishes, tending to zero, $\kappa_{z,n}^{(c)}$ grow monotonously and become very large, going to $+\infty$ [12]. This hinted that the latter might possess features, resembling those of the zeros $\zeta_{k,n}^{(c)}$ (new families of real numbers might be advanced, linked with $\kappa_{z,n}^{(c)}$).

Following the above approach, here new auxiliary numbers $D_1(c, n, z)$ are introduced through the relation: $D_1(c, n, z) = \kappa_{z,n}^{(c)} z$. The sequences $\{\kappa_{z,n}^{(c)}\}$ and $\{D_1(c, n, z)\}$ are constructed and the dependence of their terms on variable z is studied in a wide interval of variation of this parameter. It is found out that the first is divergent both for $z \to 0$ and $z \to +\infty$. In the first case it holds $\lim_{z \to 0} \kappa_{z,n}^{(c)} = +\infty$ and in the second one $\lim_{z \to +\infty} \kappa_{z,n}^{(c)} = -\infty$. As for the second set of sequences when the parameter mentioned acquires large positive values, they are divergent and tend to $-\infty$, obviously faster than $\{\kappa_{z,n}^{(c)}\}$. If $z \to 0$, however, $\{D_1(c, n, z)\}$ is convergent and possesses a finite positive real limit, depending on the numerical equivalents of c and n. The set of these limits is regarded as a family of new finite positive real numbers, called $G_1(c, n)$ ones. This is the basic results of the investigation, stated as a Theorem for existence of $G_1(c, n)$ numbers. The latter is proved numerically for positive integer c and natural n. Examples for the substantiation of assertion are given for suitably picked up (without lost of generality) and arbitrarily changing z. The work is richly illustrated by detailed tables and graphs, revealing the influence of parameters on quantities examined. The application of the new numbers in the theory of azimuthally magnetized circular ferrite waveguides, propagating normal TE_{0n} modes is demonstrated.

2. THEOREM FOR EXISTENCE OF THE $G_1(C, N)$ NUMBERS

***Theorem 1*:** *If $\kappa^{(c)}_{z,n}$ is the nth zero ($n = 1, 2, 3 \ldots$) in the imaginary part κ of the complex first parameter a of the complex Kummer confluent hypergeometric function $\Phi(a, c; x)$ with $a = c/2+j\kappa$ — complex, $c = 2\mathrm{Re}a$ — restricted positive integer ($c = 1, 2, 3 \ldots$), κ — real (positive, negative or zero, $-\infty < \kappa < +\infty$), $x = jz$ — positive purely imaginary, z — real, positive and if $D_1(c, n, z) = \kappa^{(c)}_{z,n} z$, then the infinite sequence of real numbers $\{\kappa^{(c)}_{z,n}\}$ is divergent and grows unlimitedly (tends to $+\infty$) in case $z \to 0$, whereas on the same condition the sequence $\{D_1(c, n, z)\}$ is convergent (c, n — fixed). Its limit equals the finite real positive number G_1 where $G_1 = G_1(c, n)$. It holds:*

$$G_1(c, n) = \lim_{z\to 0} D_1(c, n, z). \tag{1}$$

Stipulating that $z \to +\infty$ both $\{\kappa^{(c)}_{z,n}\}$ and $\{D_1(c, n, z)\}$ are divergent and go to $-\infty$.

***Numerical proof*:** The proof of Theorem 1 is illustrated in Table 1 for $c = 1$ and $n = 1$ (1) 4, and in Tables 2–4 for $c = 2$ (1) 7 and $n = 1$ (cf. the digits marked by bold face type). Tables 1 and 2 contain results for $z = 1\cdot10^{-s}$, $s = -1$ (1) 8, Table 3 — for z varying from $1\cdot10^{-4}$ to $1\cdot10^{-7}$, and Table 4 — for z taking in values from $1.0916527843\cdot10^{1}$ to $1.4738592601\cdot10^{-8}$. The numerical equivalents of z in Table 3 are specially chosen and in Table 4 are arbitrary. The dependence of $D_1(c, n, z) = \kappa^{(c)}_{z,n} z$ on z is portrayed with blue solid lines in Figures 1–4 for selected values of

Table 1: Numbers $\kappa^{(c)}_{z,n}$ and $D_1(c, n, z)$ for $c = 1$ and $n = 1$, 2, 3 and 4 in case of small $z = 1\cdot10^{-s}$, $s = -1$ (1) 8.

z	$\kappa^{(c)}_{z,n}$	$D_1(c,n,z)$	$\kappa^{(c)}_{z,n}$	$D_1(c,n,z)$
	$n=1$		$n=2$	
$1\cdot10^{1}$	(−1) −7.648271756996514	−7.648271756996514	(−1) 1.22948 32177 89881	1.22948 32177 89881
$1\cdot10^{0}$	(0) 1.39096 86145 43394	**1.**39096 86145 43394	(0) 7.54004 45591 61164	**7.**54004 45591 61164
$1\cdot10^{-1}$	(1) 1.44525 13179 65011	**1.445**25 13179 65011	(1) 7.61703 69580 49011	**7.617**03 69580 49011
$1\cdot10^{-2}$	(2) 1.44579 10393 18058	**1.44579** 10393 18058	(2) 7.61780 77995 46547	**7.6178**0 77995 46547
$1\cdot10^{-3}$	(3) 1.44579 64362 22541	**1.44579 64**362 22541	(3) 7.61781 55080 51823	**7.61781 55**080 51823
$1\cdot10^{-4}$	(4) 1.44579 64901 91555	**1.44579 6490**1 91555	(4) 7.61781 55851 36884	**7.61781 5585**1 36884
$1\cdot10^{-5}$	(5) 1.44579 64907 31245	**1.44579 64907 3**1245	(5) 7.61781 55859 07735	**7.61781 55859** 07735
$1\cdot10^{-6}$	(6) 1.44579 64907 36642	**1.44579 64907 366**42	(6) 7.61781 55859 15444	**7.61781 55859 15**444
$1\cdot10^{-7}$	(7) 1.44579 64907 36696	**1.44579 64907 3669**6	(7) 7.61781 55859 15521	**7.61781 55859 15521**
$1\cdot10^{-8}$	(8) 1.44579 64907 36696	**1.44579 64907 36696**	(8) 7.61781 55859 15521	**7.61781 55859 15521**
	$n=3$		$n=4$	
$1\cdot10^{1}$	(−1) 11.23870 61195 42211	**1**1.23870 61195 42211	(−1) 26.89746 05255 44971	26.89746 05255 44971
$1\cdot10^{0}$	(0) 18.64070 20791 32000	**18.**64070 20791 32000	(0) 34.67797 16513 50657	**34.**67797 16513 50657
$1\cdot10^{-1}$	(1) 18.72094 06258 98684	**18.72**094 06258 98684	(1) 34.75924 97637 33569	**34.7**5924 97637 33569
$1\cdot10^{-2}$	(2) 18.72174 35868 98565	**18.7217**4 35868 98565	(2) 34.76006 28931 51317	**34.7600**6 28931 51317
$1\cdot10^{-3}$	(3) 18.72175 16165 66037	**18.72175 16**165 66037	(3) 34.76007 10244 80322	**34.76007 1**0244 80322
$1\cdot10^{-4}$	(4) 18.72175 16968 62717	**18.72175 169**68 62717	(4) 34.76007 11057 93614	**34.76007 110**57 93614
$1\cdot10^{-5}$	(5) 18.72175 16976 65684	**18.72175 16976** 65684	(5) 34.76007 11066 06746	**34.76007 11066** 06746
$1\cdot10^{-6}$	(6) 18.72175 16976 73716	**18.72175 16976 737**16	(6) 34.76007 11066 14882	**34.76007 11066 14**882
$1\cdot10^{-7}$	(7) 18.72175 16976 73795	**18.72175 16976 73795**	(7) 34.76007 11066 14963	**34.76007 11066 14963**
$1\cdot10^{-8}$	(8) 18.72175 16976 73795	**18.72175 16976 73795**	(8) 34.76007 11066 14962	**34.76007 11066 14963**

Table 2: Numbers $\kappa_{z,n}^{(c)}$ and $D_1(c,n,z)$ for $c=2$ and 3, and $n=1$ in case of small $z = 1 \cdot 10^{-s}$, $s = -1\ (1)\ 8$.

z	$\kappa_{z,n}^{(c)}$	$D_1(c,n,z)$	$\kappa_{z,n}^{(c)}$	$D_1(c,n,z)$
	$c=2$		$c=3$	
$1{\cdot}10^{1}$	(−1) −6.985596396425155	−6.985596396425155	(−1) 5.26046 84863 74887	5.26046 84863 74887
$1{\cdot}10^{0}$	(0) 3.58691 89372 16388	**3.**58691 89372 16388	(0) 6.49118 71834 61350	**6.**49118 71834 61350
$1{\cdot}10^{-1}$	(1) 3.66965 93031 82381	**3.6**6965 93031 82381	(1) 6.59263 11796 40417	**6.59**263 11796 40417
$1{\cdot}10^{-2}$	(2) 3.67048 43271 95238	**3.6704**8 43271 95238	(2) 6.59364 38776 93587	**6.5936**4 38776 93587
$1{\cdot}10^{-3}$	(3) 3.67049 25771 97640	**3.67049 2**5771 97640	(3) 6.59365 40044 99892	**6.59365 4**0044 99892
$1{\cdot}10^{-4}$	(4) 3.67049 26596 97640	**3.67049 26**596 97640	(4) 6.59365 41057 67938	**6.59365 410**57 67938
$1{\cdot}10^{-5}$	(5) 3.67049 26605 22640	**3.67049 26605** 22640	(5) 6.59365 41067 80618	**6.59365 41067** 80618
$1{\cdot}10^{-6}$	(6) 3.67049 26605 30890	**3.67049 26605 30**890	(6) 6.59365 41067 90746	**6.59365 41067 90**746
$1{\cdot}10^{-7}$	(7) 3.67049 26605 30973	**3.67049 26605 30973**	(7) 6.59365 41067 90847	**6.59365 41067 9084**7
$1{\cdot}10^{-8}$	(8) 3.67049 26605 30973	**3.67049 26605 30973**	(8) 6.59365 41067 90848	**6.59365 41067 9084**8

Table 3: Numbers $\kappa_{z,n}^{(c)}$ and $D_1(c,n,z)$ for $c=4$ and 5, and $n=1$ in case of small specially chosen $z = \langle 1 \cdot 10^{-4} \div 1 \cdot 10^{-7} \rangle$.

z	$\kappa_{z,n}^{(c)}$	$D_1(c,n,z)$	$\kappa_{z,n}^{(c)}$	$D_1(c,n,z)$
	$c=4$		$c=5$	
1.10^{-4}	(4) 10.17661 64533 89198	**10.17661 645**33 89198	(4) 14.39573 52245 55291	**14.39573 522**45 55291
8.10^{-5}	(5) 1.27207 70567 25889	**10.17661 645**38 07115	(5) 1.79946 69031 26448	**14.39573 5225**0 11587
6.10^{-5}	(5) 1.69610 27423 55360	**10.17661 6454**1 32162	(5) 2.39928 92042 27747	**14.39573 5225**3 66485
4.10^{-5}	(5) 2.54415 41135 91085	**10.17661 6454**3 64338	(5) 3.59893 38064 04995	**14.39573 5225**6 19982
2.10^{-5}	(5) 5.08830 82272 51822	**10.17661 64545** 03644	(5) 7.19786 76128 86040	**14.39573 5225**7 72080
1.10^{-5}	(5) 10.17661 64545 38470	**10.17661 64545** 38470	(5) 14.39573 52258 10105	**14.39573 52258** 10105
8.10^{-6}	(6) 1.27207 70568 17831	**10.17661 64545** 42651	(6) 1.79946 69032 26834	**14.39573 52258** 14670
6.10^{-6}	(6) 1.69610 27424 24317	**10.17661 64545** 45901	(6) 2.39928 92043 03036	**14.39573 52258** 18218
4.10^{-6}	(6) 2.54415 41136 37056	**10.17661 64545** 48223	(6) 3.59893 38064 55188	**14.39573 52258 2**0754
2.10^{-6}	(6) 5.08830 82272 74808	**10.17661 64545** 49616	(6) 7.19786 76129 11137	**14.39573 52258 22**275
1.10^{-6}	(6) 10.17661 64545 49964	**10.17661 64545** 49964	(6) 14.39573 52258 22655	**14.39573 52258 22**655
8.10^{-7}	(7) 1.27207 70568 18751	**10.17661 64545 500**06	(7) 1.79946 69032 27838	**14.39573 52258 227**01
6.10^{-7}	(7) 1.69610 27424 25006	**10.17661 64545 500**39	(7) 2.39928 92043 03789	**14.39573 52258 227**36
4.10^{-7}	(7) 2.54415 41136 37515	**10.17661 64545 500**62	(7) 3.59893 38064 55690	**14.39573 52258 227**62
2.10^{-7}	(7) 5.08830 82272 75038	**10.17661 64545 5007**6	(7) 7.19786 76129 11388	**14.39573 52258 2277**7
1.10^{-7}	(7) 10.17661 64545 50079	**10.17661 64545 50079**	(7) 14.39573 52258 22781	**14.39573 52258 2278**1

Table 4: Numbers $\kappa_{z,n}^{(c)}$ and $D_1(c,n,z)$ for $c=6$ and 7, and $n=1$ in case of small arbitrary $z = \langle 1.0916527843 \cdot 10^{1} \div 1.4738592601 \cdot 10^{-8} \rangle$.

z	$\kappa_{z,n}^{(c)}$	$D_1(c,n,z)$	$\kappa_{z,n}^{(c)}$	$D_1(c,n,z)$
	$c=6$		$c=7$	
$1.0916527843{\cdot}10^{1}$	(−1) 1.91110 05127 17661	2.0862581957 85393	(−1) 6.30276 35571 41752	6.880429 38593 83663
$6.4579803216{\cdot}10^{0}$	(0) 2.08374 96909 78616	**1**3.45681 44994 79985	(0) 2.88536 95948 94069	18.63366 00643 68859
$5.4698720315{\cdot}10^{-1}$	(1) 3.50908 33885 10577	**19.**19423 70830 15252	(1) 4.50448 40337 16057	**24.6**3895 12323 61765
$4.9675802134{\cdot}10^{-2}$	(2) 3.87198 54180 50890	**19.234**39 81492 82930	(2) 4.96845 86081 62088	**24.681**21 66730 02891
$9.7685401239{\cdot}10^{-3}$	(3) 1.96904 74652 68049	**19.2347**1 91703 34529	(3) 2.52663 69606 91400	**24.6815**5 45290 42686
$8.7964502138{\cdot}10^{-4}$	(4) 2.18664 70577 55958	**19.23473** 19787 02541	(4) 2.80585 54768 36671	**24.68156 8**0091 11842
$3.7946508213{\cdot}10^{-5}$	(5) 5.06890 69927 72371	**19.23473 2083**2 16992	(5) 6.50430 54766 89861	**24.68156 8119**1 07269
$7.3495602817{\cdot}10^{-6}$	(6) 2.61712 69227 21889	**19.23473 20834** 04539	(6) 3.35823 73874 47573	**24.68156 81193** 04654
$2.1743590628{\cdot}10^{-7}$	(7) 8.84616 18011 89630	**19.23473 20834 1184**3	(7) 11.35119 24233 56703	**24.68156 81193 1234**1
$1.4738592601{\cdot}10^{-8}$	(8) 13.05058 94315 21979	**19.23473 20834 11849**	(8) 16.74621 77614 14835	**24.68156 81193 12347**

parameters c and n. The limiting $G_1(c,n)$ numbers to which the $D_1(c,n,z)$ ones (the curves) tend provided z gets infinitesimal, are depicted in by red circles at the left vertical (zero) axes of the figures mentioned. The effect of c and n on $G_1(c,n)$ is presented in Table 5 and Figure 5. In the latter the quantities studied for the specific n are pictured by the same and for different c — by distinct symbols. The statement of Theorem 1 concerning the impact z on zeros $\kappa_{z,n}^{(c)}$ of $\Phi(a, c; x)$

Table 5: Dependence of the $G_1(c, n)$ numbers on $c = 1$ (1) 10 and $n = 1$ (1) 5.

$c \backslash n$	1	2	3	4	5
1	1.44579 64907 36696	7.61781 55859 15521	18.72175 16976 73795	34.76007 11066 14962	55.73307 59044 08538
2	3.67049 26605 30973	12.30461 40804 23651	25.87486 34737 84145	44.38019 17034 51162	67.82041 35682 18332
3	6.59365 41067 90848	17.71249 97297 73964	33.75517 72164 92605	54.73004 72864 15859	80.63877 90731 36191
4	10.17661 64545 50079	23.81939 31360 09286	42.34886 24565 24859	65.80021 35637 52044	94.18134 98580 41812
5	14.39573 52258 22781	30.60694 90162 32036	51.64245 25942 48102	77.58056 49669 69119	108.44028 40984 51510
6	19.23473 20834 11849	38.06028 83854 37220	61.62386 65333 12550	90.06137 04799 38087	123.40765 80609 09464
7	24.68156 81193 12347	46.16720 18347 92032	72.28247 08239 01675	103.23361 85655 50203	139.07584 47505 93669
8	30.72690 00509 04058	54.91750 16695 84654	83.60892 14637 20086	117.08907 44284 45232	155.43765 71908 63442
9	37.36322 02158 56614	64.30255 02494 75972	95.59497 37577 97756	131.62024 25129 25234	172.48638 78484 82913
10	44.58433 53104 07377	74.31492 00693 89638	108.23330 94261 80086	146.82029 86714 54876	190.21580 17213 53713

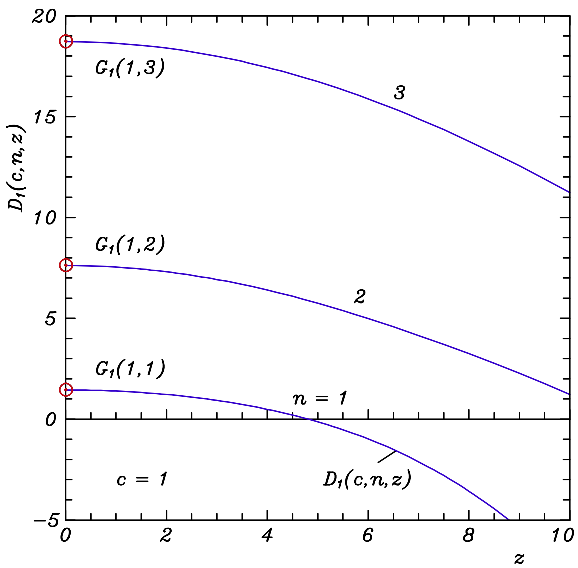

Figure 1: Graphical representation of the $D_1(c, n, z) = \kappa_{z,n}^{(c)} z$ numbers vs. z and of the $G_1(c, n)$ numbers for $c = 1$ and $n = 1$ (1) 3.

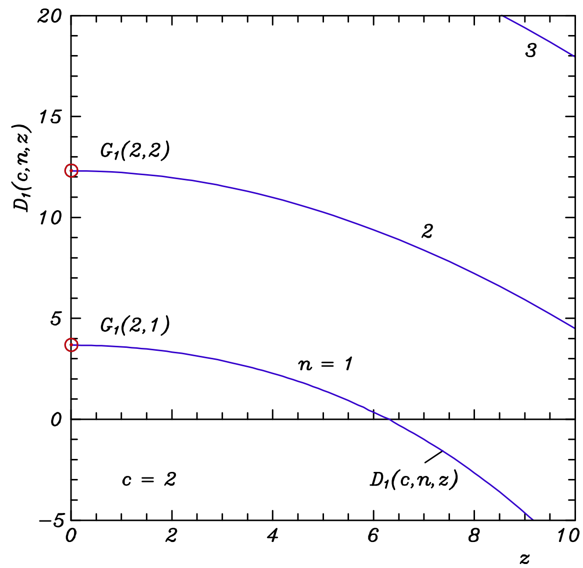

Figure 2: Graphical representation of the $D_1(c, n, z) = \kappa_{z,n}^{(c)} z$ numbers vs. z and of the $G_1(c, n)$ numbers for $c = 2$ and $n = 1, 2$.

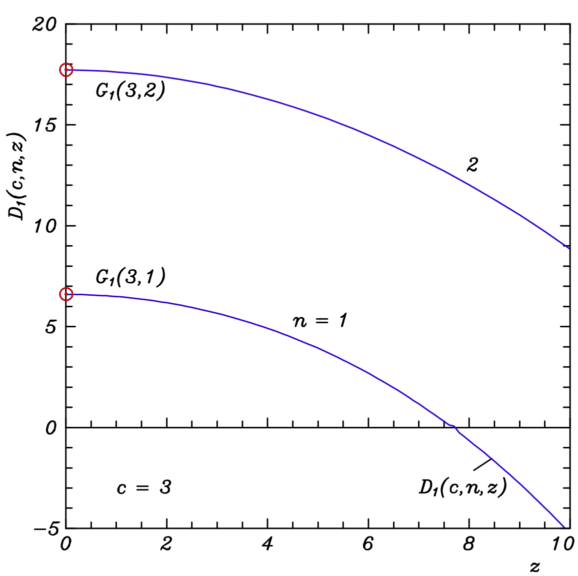

Figure 3: Graphical representation of the $D_1(c, n, z) = \kappa_{z,n}^{(c)} z$ numbers vs. z and of the $G_1(c, n)$ numbers for $c = 3$ and $n = 1$ (1) 3.

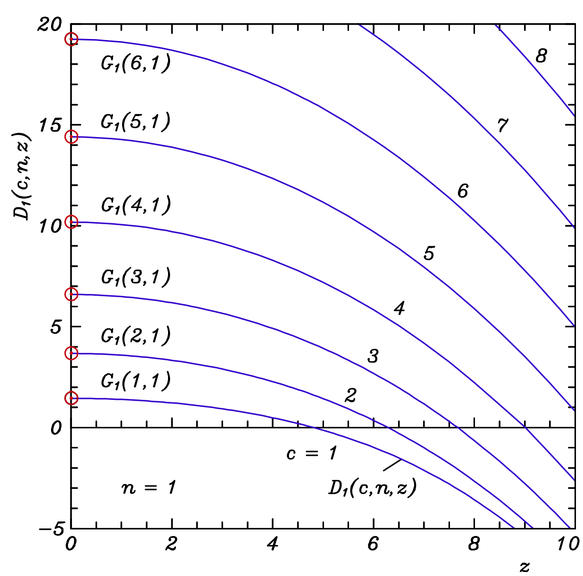

Figure 4: Graphical representation of the $D_1(c, n, z) = \kappa_{z,n}^{(c)} z$ numbers vs. z and of the $G_1(c, n)$ numbers for $c = 1$ (1) 8 and $n = 1$.

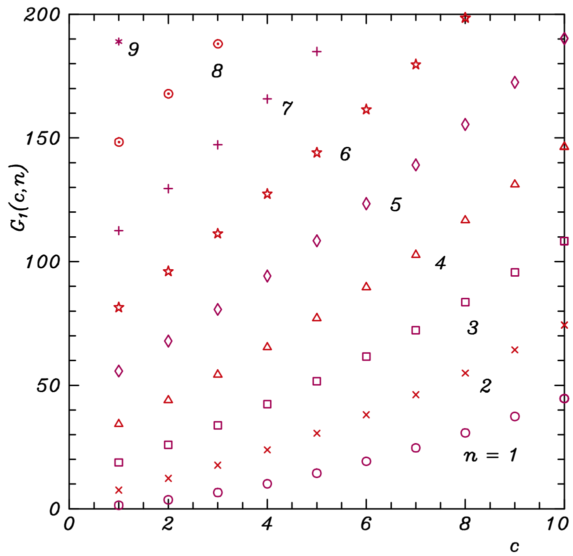

Figure 5: Dependence of the $G_1\,(c, n)$ numbers on $c = 1$ (2) 10 and $n = 1$ (1) 9.

is supported by the data in the first columns of Tables 1–4, presented here and in Tables 1(a), 1(b) and Figures 1–4 in Ref. [12]. The variation of $D_1(c, n, z)$ with z is demonstrated in Tables 1–4 and in Figures 1–4.

3. APPLICATION

3.1. Propagation Problem for Normal TE_{0n} Modes in the Azimuthally Magnetized Circular Ferrite Waveguide

The transmission of normal TE_{0n} modes of phase constant β in the circular waveguide of radius r_0, uniformly filled with azimuthally magnetized lossless remanent ferrite, described by a scalar permittivity $\varepsilon = \varepsilon_0\varepsilon_r$ and a Polder permeability tensor of off-diagonal element $\alpha = \gamma M_r/\omega$, $0 < |\alpha| < 1$, (γ — gyromagnetic ratio, M_r — remanent magnetization, ω — angular frequency of the wave), is governed by the equation [2, 3]:

$$\Phi\,(a, c; x_0) = 0, \tag{2}$$

in which $a = c/2 - jk$, $c = 3$, $x_0 = jz_0$, $k = \alpha\bar{\beta}/(2\bar{\beta}_2)$, $z_0 = 2\bar{\beta}_2\bar{r}_0$, $\bar{\beta} = \beta/(\beta_0\sqrt{\varepsilon_r})$, $\bar{\beta}_2 = \beta_2/(\beta_0\sqrt{\varepsilon_r})$, $\bar{r}_0 = \beta_0 r_0\sqrt{\varepsilon_r}$, $\beta_0 = \omega\sqrt{\varepsilon_0\mu_0}$, $\beta_2 = [\omega^2\varepsilon_0\mu_0\varepsilon_r(1-\alpha^2) - \beta^2]^{1/2}$ — radial wavenumber, i.e., $\bar{\beta}_2 = [(1-\alpha^2) - \bar{\beta}^2]^{1/2}$, ($k$, z_0 — real, $-\infty < k < +\infty$, $z_0 > 0$).

3.2. Traditional Solution in Terms of the Zeros $\zeta^{(c)}_{k,n}$ of $\Phi(a, c; x)$ and the $L_1(c, n)$ Numbers

The approach for finding the phase portrait of the structure which historically first has been developed, uses the roots (zeros) $\zeta^{(c)}_{k,n}$ ($n = 1, 2, 3$) in z_0 of Eq. (2) (of the Kummer function) with the values of its parameters, given in Subsection 3.1 [2, 3, 5]. In particular, the eigenvalue spectrum is determined from the term [2, 3, 5]:

$$\bar{\beta}_2 = \zeta^{(c)}_{k,n}/\,(2\bar{r}_0) \tag{3}$$

and the $\bar{\beta}(\bar{r}_0)$ — phase characteristics are computed by means of the expressions [2, 3, 5]:

$$\bar{r}_0 = \left(k\zeta^{(c)}_{k,n}/\alpha\right)\left\{\left[1 + (\alpha/\,(2k))^2\right]/\left(1-\alpha^2\right)\right\}^{1/2}, \tag{4}$$

$$\bar{\beta} = \left\{\left(1-\alpha^2\right)/\left[1 + (\alpha/\,(2k))^2\right]\right\}^{1/2}, \tag{5}$$

in that α is a discrete and k — a varying parameter. For the purpose a special iterative technique has been elaborated [2, 5]. A peculiarity of the configuration considered is the appearance of cut-off for higher frequencies in case of negative (clockwise) ferrite magnetization in addition to the conventional ones. They lie on a special En_{1-} — envelope line of equation $\bar{\beta}_{en-} = \bar{\beta}_{en-}(\bar{r}_{0en-})$, presented in parametric form as [2,5]:

$$\bar{r}_{0en-} = L_1\left(c,n\right) / \left[\left|\alpha_{en-}\right|\left(1-\alpha_{en-}^2\right)^{1/2}\right], \tag{6}$$

$$\bar{\beta}_{en-} = \left(1-\alpha_{en-}^2\right)^{1/2} \tag{7}$$

with α_{en-} is a parameter. (The subscript "$en-$" distinguishes the quantities, relevant to the envelope.)

3.3. New Solution in Terms of the Zeros $\kappa_{z,n}^{(c)}$ of $\Phi(a,c;x)$ and the $G_1(c,n)$ Numbers

Quite recently a novel line of attack to the task in question has been advanced, employing It is suggested to expresses the roots $\kappa_{z,n}^{(c)}$ ($n = 1,\ 2,\ 3$) in κ of Eq. (2) (of the Kummer function) in which its first parameter is rewritten in the form $a = c/2 + j\kappa$ ($\kappa = -k$) [12]. Accordingly, Formulae (3)–(5) are modified as follows [12]:

$$\bar{\beta}_2 = \left\{\left(1-\alpha^2\right) / \left[1+\left(2\kappa_{z,n}^{(c)}/\alpha\right)^2\right]\right\}^{1/2}. \tag{8}$$

$$\bar{r}_0 = \left(\kappa_{z,n}^{(c)}\, z/\alpha\right)\left\{\left[1+\left(\alpha/\left(2\kappa_{z,n}^{(c)}\right)\right)^2\right] / \left(1-\alpha^2\right)\right\}^{1/2}, \tag{9}$$

$$\bar{\beta} = \left\{\left(1-\alpha^2\right) / \left[1+\left(\alpha/\left(2\kappa_{z,n}^{(c)}\right)\right)^2\right]\right\}^{1/2}, \tag{10}$$

where α is a discrete and z — a varying parameter.

Applying the statement of Theorem 1 for $z \to 0$ ($\kappa_{z,n}^{(c)}\, z \to G_1(c,n)$ and $\kappa_{z,n}^{(c)} \to +\infty$) a new representation of the En_{1-} — line is obtained:

$$\bar{r}_{0en-} = G_1\left(c,n\right) / \left[\left|\alpha_{en-}\right|\left(1-\alpha_{en-}^2\right)^{1/2}\right], \tag{11}$$

$$\bar{\beta}_{en-} = \left(1-\alpha_{en-}^2\right)^{1/2}. \tag{12}$$

4. CONCLUSION

The finite real positive numbers $G_1(c,n)$ are advanced for c — positive integer and n — bounded natural number as limits of special sequences of real numbers. The terms of the latter are devised as products of the zeros $\kappa_{z,n}^{(c)}$ of the complex Kummer function $\Phi(a,c;x)$ in the imaginary part κ of its complex first parameter $a = c/2 + j\kappa$ by the imaginary part z of its positive purely imaginary variable $x = jz$ ($n = 1,\ 2,\ 3,\ \ldots$ — numbers of the zeros). The limits in question are attained for infinitesimal z. This constitutes the essence of the formulated and numerically proved here Theorem for existence of $G_1(c,n)$ numbers being the core of the study. The usage of some of their representatives is also considered.

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