Nonlinear Goubau Line: Numerical Study of TE-polarized Waves

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Abstract— The propagation of monochromatic electromagnetic TE waves in the Goubau line (conducting cylinder covered by a concentric dielectric layer) filled with nonlinear inhomogeneous medium is considered. Two types of nonlinearity are studied. A physical problem is reduced to solving a nonlinear transmission eigenvalue problem for an ordinary differential equation. Eigenvalues of the problem correspond to propagation constants of the waveguide. A method is proposed for finding approximate eigenvalues of the nonlinear problem based on solving an auxiliary Cauchy problem (by the shooting method). The existence of eigenvalues that correspond to a new propagation regime is predicted. A comparison with the linear case is given.

1. INTRODUCTION

Analysis of wave propagation in open metal-dielectric waveguides constitutes an important class of vector electromagnetic problems. A conducting cylinder covered by a concentric dielectric layer, the Goubau line (GL) shown in Fig. 1 is the simplest type of such guiding structures. A complete mathematical investigation of the spectrum of symmetric surface modes in a GL is performed in [1]. Papers [2, 3] develop the theory of wave propagation in layered nonlinear dielectric waveguides. However, there are virtually no results concerning the analysis of wave propagation in a GL with an external concentric layer having variable, nonlinear, or variable and nonlinear permittivity. We aim this study to fill this gap and consider the problem of electromagnetic TE (transverse-electric) wave propagation in a GL with nonlinear inhomogeneous permittivity of the dielectric medium filling the GL layer. We consider only the intensity-dependent permittivity. The determination of TE waves is reduced to a nonlinear transmission eigenvalue problem for Maxwell's equations, which is then reduced to a system of nonlinear ordinary differential equations. Numerical experiments are carried out for two types of nonlinearities.

2. STATEMENT OF THE PROBLEM

Consider three-dimensional space $\mathbb{R}^3$ with a cylindrical coordinate system $O\rho\varphi z$ filled with isotropic medium with constant permittivity $\varepsilon = \varepsilon_c\varepsilon_0$, ($\varepsilon_0 > 0$ is the permittivity of free space and we assume $\mu = \mu_0$, where $\mu_0 > 0$ is magnetic permeability of vacuum). A GL with a cross-section

$$\Sigma := \{(\rho, \varphi, z) : a \leqslant \rho \leqslant b, 0 \leq \varphi < 2\pi\}$$

is placed parallel to the axis Oz (Fig. 1). We will consider TE-polarized waves in the monochromatic mode

$$\mathbf{E}e^{-i\omega t} = e^{-i\omega t}\left(0, E_\varphi, 0\right)^T,$$
$$\mathbf{H}e^{-i\omega t} = e^{-i\omega t}\left(H_\rho, 0, H_z\right)^T,$$

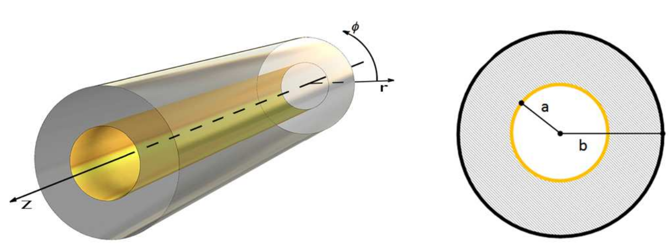

Figure 1: The Goubau line, where a and b are the radii of the internal (perfectly conducting) and external (dielectric) cylinders, respectively.

where $(\cdot)^T$ denotes the transpose operation. Each component of the field $\mathbf{E}$, $\mathbf{H}$ is a function of three spatial variables. Complex amplitudes of the electromagnetic field $\mathbf{E}$, $\mathbf{H}$ satisfy the Maxwell equations

$$\begin{cases} \mathrm{rot}\mathbf{H} = -\mathrm{i}\omega\varepsilon\mathbf{E}, \\ \mathrm{rot}\mathbf{E} = \mathrm{i}\omega\mu\mathbf{H}, \end{cases} \tag{1}$$

the continuity condition for the tangential components on the media interfaces (on the boundary of the waveguide) $\rho = a$, $\rho = b$ and the radiation condition at infinity: the electromagnetic field exponentially decays as $O(|\rho|^{-1})$ when $\rho \to \infty$. The solutions to the Maxwell equations are sought in the entire space. The dielectric permittivity in the whole space has the form $\varepsilon = \widetilde{\varepsilon}\varepsilon_0$, where

$$\widetilde{\varepsilon} = \begin{cases} \varepsilon(\rho) + \alpha f\left(|\mathbf{E}|^2\right), & a \leqslant \rho \leqslant b, \\ \varepsilon_c, & \rho > b, \end{cases} \tag{2}$$

and ε_c — positive real constant, $\varepsilon(\rho) \in C^1[R_1, R_2]$, $\varepsilon(\rho) > 0$, α is a real constants.

Remark 1. *Inside the waveguide Σ quantities ε_c and $\varepsilon(\rho) + \alpha f\left(|\mathbf{E}|^2\right)$ are scalars. It is possible to consider the case where one of these quantities (or both of them) is a diagonal tensor. Such generalization will only lead to complication of formulas.*

The surface waves propagating along the axis Oz of the waveguide Σ have the form

$$E_\varphi = \mathrm{E}_\varphi(\rho)e^{i\gamma z}, \quad H_\rho = \mathrm{H}_\rho(\rho)e^{i\gamma z}, \quad H_z = \mathrm{H}_z(\rho)e^{i\gamma z}, \tag{3}$$

where γ is the propagation constant (spectral parameter of the problem). In what follows we often omit the arguments of functions when it does not lead to misunderstanding.

Let $k_0^2 := \omega^2\mu_0\varepsilon_0$. Substituting complex amplitude $\mathbf{E}$ and $\mathbf{H}$ with components (3) into equations of system (1) and introducing the notation $u(\rho) := \mathrm{E}_\varphi(\rho)$, we obtain

$$\left(\rho^{-1}\left(\rho u\right)'\right)' + \left(k_0^2\widetilde{\varepsilon} - \gamma^2\right)u = 0 \tag{4}$$

where u is a sufficiently smooth real function,

$$u \in C^1\left[0, +\infty\right) \cap C^2\left(0, R_1\right) \cap C^2\left(R_1, R_2\right) \cap C^2\left(R_2, +\infty\right). \tag{5}$$

The tangential components E_φ and H_z are continuous at the interface:

$$[u]|_{\rho=a} = [u]|_{\rho=b} = \left[u'\right]\big|_{\rho=b} = 0, \tag{6}$$

where $[v]|_{\rho=s} = \lim\limits_{\rho\to s-0} v(\rho) - \lim\limits_{\rho\to s+0} v(\rho)$ is the jump of the limit values of the function at a point s defined by (5).

Formulate

Problem P_E: Find quantities $\hat{\gamma}$ such that for a given field on one of the boundaries of the waveguide, there exists a non-trivial function $u(\rho;\hat{\gamma})$ such that for $\rho \in [0, +\infty]$ $u(\rho;\hat{\gamma})$ satisfies the equation (4), transmission conditions (5), (6), and the radiation condition at infinity: functions $u(\rho;\hat{\gamma})$ exponentially decay as $\rho \to \infty$.

The quantities solving problem P_E are called eigenvalues and the corresponding $u(\rho;\hat{\gamma})$ eigenfunctions. It should be noted that an eigenvalue γ depends on the value of the eigenfunction on one of the waveguide boundaries.

3. DISPERSION EQUATION

Taking into account the radiation condition at infinity, we show that the solution and the ?rst derivative of solution of Equation (4) in the domain $\rho > b$ have the form, respectively,

$$u(\rho) = CK_1\left(k\rho\right), \tag{7}$$

$$u'(\rho) = -Ck\left(K_0\left(k\rho\right) + \frac{K_1\left(k\rho\right)}{k\rho}\right), \tag{8}$$

where $k^2 := \gamma^2 - k_0^2\varepsilon_c$ and K_0, K_1 are the modified Bessel functions. Constant C assumed to be known. Solution (7) are real, which yields $\gamma^2 > k_0^2\varepsilon_c$.

Remark 2. *As was noted above the eigenvalues depend on the value of the field on the boundary of the waveguide; for this reason, the constant C is considered to be fixed (and known in calculations).*

Consider the Cauchy problem for

$$u'' = -\rho^{-1}u' + \rho^{-2}u - \left(k_0^2\varepsilon\left(\rho\right) - \gamma^2\right)u - \alpha f\left(|u|^2\right)u \tag{9}$$

with the initial conditions $u(b) := u(b+0)$, $u'(b) := u'(b+0)$, where the values $u(b+0)$ and $u'(b+0)$ are determined from (7) and (8).

Choosing the value of the constant $C = \frac{1}{K_1(k\rho)}$ we obtain initial conditions in the following form

$$u(b) = 1, \tag{10}$$

$$u'(b) = -k\frac{K_0\left(kb\right)}{K_1\left(kb\right)} - \frac{1}{b}. \tag{11}$$

Note that, unlike problems for linear media, in the case of nonlinear media, the propagation constants (eigenvalues) depend on the field amplitude.

To justify the solution technique, we use classical results of the theory of ordinary differential equations concerning the existence and uniqueness of the solution to the Cauchy problem and continuous dependence of the solution on parameters.

Using the transmission condition on the boundary $\rho = a$ (see (6)) and the fact that $u(a-0;\gamma) = 0$ ($\mathbf{E} = 0$ for $\rho < a$) we obtain the following dispersion equation

$$\Delta\left(\gamma\right) \equiv u\left(a+0;\gamma\right) = 0, \tag{12}$$

where $\Delta(\gamma)$ is determined explicitly and quantity $u(a+0;\gamma)$ is obtained from the solution to the Cauchy problem.

Thus, when the number $\gamma = \tilde{\gamma}$ is such that $\Delta(\gamma) = 0$, then γ is the propagation constant for problem P_E.

Remark 3. *It is known that in the linear case ($\alpha = 0$) Equation (12) has a real solution γ such that*

$$k\varepsilon_c < \gamma^2 < k\min_{\rho\in[a,b]}\varepsilon(\rho).$$

We assume that in the nonlinear case Equation (12) also has a real solution γ such that

$$k\varepsilon_c < \gamma^2 < k\min_{\rho\in[a,b]}\varepsilon(\rho) + \delta,$$

where δ is a real constant.

Equation (12) can be solved, for example, by the shooting method [9].

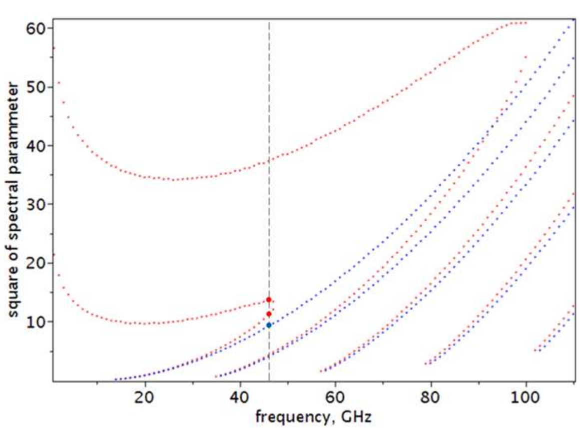

Figure 2: Red and blue curves correspond, respectively, to nonlinear ($\alpha_k = 0.25$) and linear ($\alpha_k = 0$) cases.

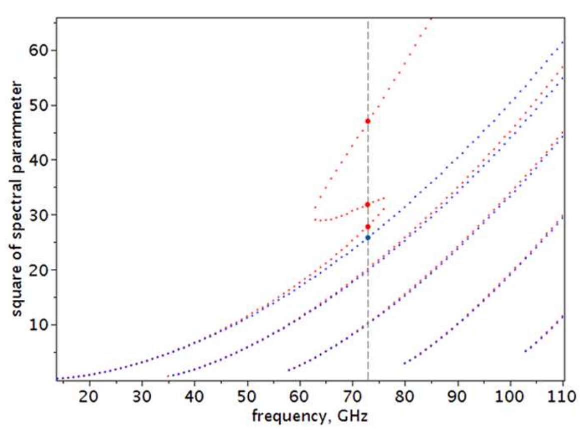

Figure 3: Red and blue curves correspond, respectively, to nonlinear ($\alpha_s = 10$, $\beta_s = 0.005$) and linear ($\alpha_s = 0$) cases.

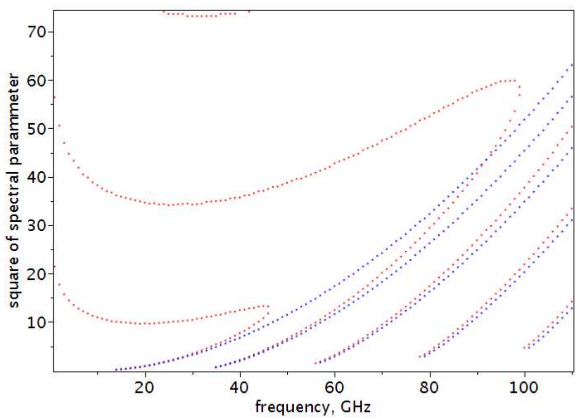

Figure 4: Red and blue curves correspond, respectively, to nonlinear ($\alpha_k = 0.25$) and linear ($\alpha_k = 0$) cases.

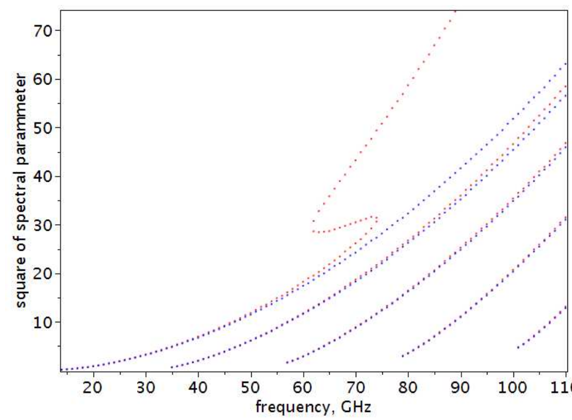

Figure 5: Red and blue curves correspond, respectively, to nonlinear ($\alpha_s = 10$, $\beta_s = 0.005$) and linear ($\alpha_s = 0$) cases.

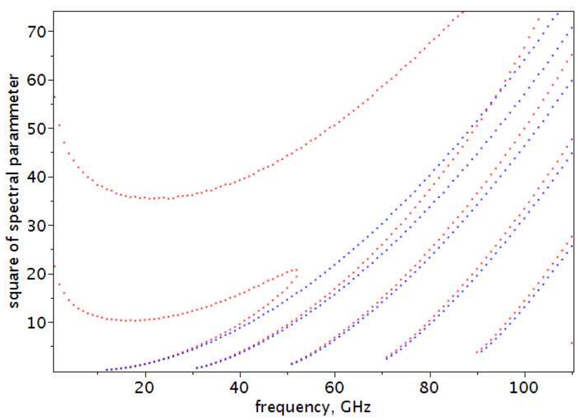

Figure 6: Red and blue curves correspond, respectively, to nonlinear ($\alpha_k = 0.25$) and linear ($\alpha_k = 0$) cases.

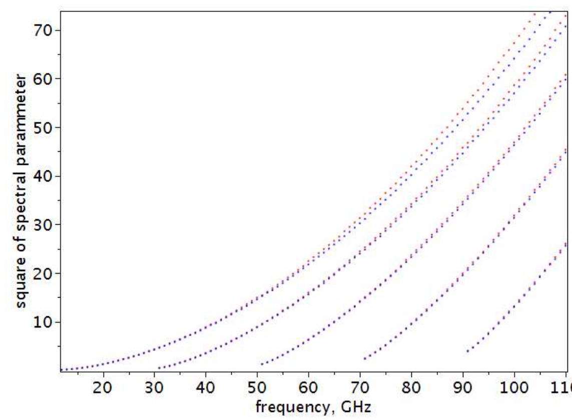

Figure 7: Red and blue curves correspond, respectively, to nonlinear ($\alpha_s = 10$, $\beta_s = 0.005$) and linear ($\alpha_s = 0$) cases.

4. NUMERICAL

Numerical experiments are carried out for two types of nonlinearities:

1. $\varepsilon = \varepsilon(\rho) + \alpha_k u^2$ (Kerr law) and
2. $\varepsilon = \varepsilon(\rho) + \alpha_s(1 - e^{-\beta_s u^2})$ (nonlinearity with saturation),

where $\varepsilon(\rho)$ is a function that specify the inhomogeneity inside GL:

1. $\varepsilon(\rho) = \varepsilon$;
2. $\varepsilon(\rho) = \varepsilon + \frac{1}{\rho}$;
3. $\varepsilon(\rho) = \varepsilon + \rho$,

where ε is a positive real constant.

In Figs. 2–7 the eigenvalues or dependence $\gamma(f)$ are shown. The following values of parameters are used in calculations: frequency $1\,\mathrm{GHz} \leqslant f \leqslant 110\,\mathrm{GHz}$, $\varepsilon_c = 1$ (vacuum), $\varepsilon = 12$ (silicon), $a = 2\,\mathrm{mm}$, $b = 4\,\mathrm{mm}$, and $\alpha_k = 0.25$ for the Kerr nonlinearity and $\alpha_s = 10$ and $\beta_s = 0.005$ for the nonlinearity with saturation.

5. CONCLUSION

The analytical approach developed in this study allows one to prove the existence of eigenvalues in the nonlinear problem that are close to eigenvalues of the corresponding linear problem.

However, as it is seen from the plots above, in the nonlinear problem, there are new eigenvalues that cannot be determined as a perturbation of eigenvalues of the linear problem. These ‘new’ eigenvalues correspond to a new (purely nonlinear) propagation regime. At the same time, the proposed numerical method enables finding all eigenvalues and proves to be efficient in the case of discrete eigenvalues. In Figures 2 and 3 the groups of eigenvalues are marked with different colors blue dots (smallest value) are eigenvalues of the linear problem (with $\alpha = 0$), and red dots are eigenvalues of the nonlinear problem. When one passes to the limit $\alpha \to 0$, the first red dot tends to the blue one, while other red dots do not tend to any eigenvalue of the linear problem when $\alpha \to 0$; these eigenvalues are 'purely nonlinear' and correspond to a new type of nonlinear waves.

Whether these mathematically predicted purely nonlinear waves really exist, is a hypothesis that can be proved or disproved in physical experiment.

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