Dissipative Magnetorotational Instability: Wavelength Asymptotic Saturation

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Abstract— When a plasma, which rotates differentially about a fixed center, is subjected to a magnetic field, perpendicular to the plane of rotation of the fluid, the coupling of the current with the field (the Lorentz force density) may be disruptive if the angular velocity of the gas decreases with the increase of the radial coordinate. This phenomenon is commonly referred to as the magnetorotational instability (MRI). For a perfectly conducting, inviscid plasma (ideal approximation), the problem can be treated analytically. Such an approach is particularly useful for the description of the dynamic evolution of accretion disks, astrophysical structures consisting of ionized gases which rotate about compact objects (black holes, neutron stars). In this case, the flow is assumed to exhibit a Keplerian profile, thereby leading to a dispersion relation which is biquadratic in the growth rate of the MRI. However, the associated instability condition implies the perturbative wavelength increases with the increase of the radial coordinate. This means that, as the radial coordinate decreases, the perturbative frequency increases with no limit. Recently, there has been some progress towards an analytical formulation of the MRI by including dissipative effects for the rotating plasma. In this work, by introducing both finite resistivity and viscosity for a Keplerian accretion disk, it is found that the growth rate of the instability may satisfy a quadratic equation and become suppressed by a term which depends on the magnetic Prandtl number. It is also shown that, when resistive effects dominate, the perturbative wavelength saturates asymptotically to a minimum value which does not depend on the radial coordinate.

1. INTRODUCTION

When a plasma rotates differentially about a fixed center, on a plane perpendicular to a magnetic field, the coupling of the current density flowing in the fluid with the field (the Lorentz force density) may be disruptive. Such a process is commonly referred to as the magnetorotational instability (MRI) [1, 2]. This phenomenon has been intensively studied in recent years, in connection with the description of the dynamic evolution of accretion disks [3, 4]. These systems are astrophysical structures which consist of ionized gases that orbit massive compact objects like black holes and neutron stars. Perhaps, the simplest approach to the problem is to consider an ideal plasma (dissipative effects are negligible) subjected to a small perturbation parallel to the equilibrium magnetic field. In this case, it is widely known that the condition for the MRI to occur in the neighborhood of a given point over the disk depends on the distance of the point from the central object [5, 6]. However, it can be shown that such a result implies the characteristic wavelength of the unstable modes is unbounded. In this work, we derive the condition for the MRI to occur in a plasma with both finite resistivity and viscosity. As a consequence, we find that the characteristic wavelength of the unstable modes saturates asymptotically to a finite value which does not depend on the distance from the central object.

2. BASIC EQUATIONS

Let us start by considering an incompressible fluid with (constant and uniform) mass density ρ_0. In this case, the principle of matter conservation (the continuity equation) implies the velocity field $\vec{V}$ is divergenceless,

$$\nabla \cdot \vec{V} = 0. \tag{1}$$

Next, by supposing that the fluid is a (electrically neutral) viscous plasma, subjected to a magnetic field $\vec{B}$, the time evolution of $\vec{V}$ will be governed by the Navier-Stokes equation,

$$\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla\right) \vec{V} = -\frac{\nabla P}{\rho_0} + \frac{\left(\nabla \times \vec{B}\right) \times \vec{B}}{\mu_0 \rho_0} + \frac{\nu}{\rho_0} \nabla^2 \vec{V}, \tag{2}$$

where P denotes the hydrodynamic pressure, ν represents the (dynamic) shear viscosity, μ_0 stands for the vacuum magnetic permeability (a diamagnetic medium is assumed), and use has been made of Ampère's law (in the hydromagnetic approximation: the displacement current is negligible with respect to the conduction current in the Ampère-Maxwell law). Further, the absence of magnetic monopoles implies

$$\nabla \cdot \vec{B} = 0. \tag{3}$$

Finally, the time evolution of $\vec{B}$ obeys Faraday's law,

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{V} \times \vec{B}\right) + \frac{\eta}{\mu_0}\nabla^2 \vec{B}, \tag{4}$$

where use has been made of the standard form of Ohm's law for a plasma with finite resistivity η. Eqs. (1) to (4) are the basic equations for the considerations to follow.

3. EQUILIBRIUM CONFIGURATION

We suppose now that the plasma rotates differentially about a fixed center, on a plane which is perpendicular to a constant and uniform magnetic field. By adopting cylindrical coordinates (r, θ, z), such hypotheses may be expressed through the relations $\vec{V} = \hat{\theta} r\Omega$ and $\vec{B} = \hat{z} B_0$, where $\Omega = \Omega(r)$ and $B_0 =$ constant. In this case, we see at once that Eqs. (1), (3), and (4) are automatically satisfied, and that Eq. (2) implies

$$\frac{1}{\rho_0}\frac{\partial P}{\partial r} = r\Omega^2, \qquad \frac{1}{\nu}\frac{\partial P}{\partial \theta} = 3r\Omega' + r^2\Omega'', \tag{5}$$

where the prime denotes a first order total derivative with respect to r. Eqs. (5) describe an equilibrium configuration (actually, a stationary state) of the fluid, for which cylindrical isobaric surfaces are supported radially by a centrifugal force and azimuthally by a viscous torque.

4. PERTURBATIVE ANALYSIS

Let us consider small perturbations $f \sim \exp(\gamma t + \imath k z)$ (expressed in terms of pure Fourier mode decompositions) for the equilibrium quantities F, $F \to F + f$, such that $\mathcal{O}\left(f^2\right), \mathcal{O}\left(f/r^2\right) \sim 0$. This means that we regard linearized disturbances for the equilibrium configuration (5) in the neighborhood of a given radial coordinate, to the first order in $1/r$ (only first order curvature terms are kept). In this case, Eq. (1) leads to

$$\frac{v_r}{r} + \imath k v_z = 0, \tag{6}$$

it follows from Eq. (2) that

$$\begin{aligned}
\left(\gamma + \frac{\nu k^2}{\rho_0}\right) v_r - 2\Omega v_\theta &= \imath \frac{k B_0}{\mu_0 \rho_0} b_r, \\
\left(\gamma + \frac{\nu k^2}{\rho_0}\right) v_\theta + \left(2\Omega + r\Omega'\right) v_r &= \imath \frac{k B_0}{\mu_0 \rho_0} b_\theta, \\
\left(\gamma + \frac{\nu k^2}{\rho_0}\right) v_z &= -\imath \frac{k}{\rho_0} p,
\end{aligned} \tag{7}$$

Eq. (3) leads to

$$\frac{b_r}{r} + \imath k b_z = 0, \tag{8}$$

and it follows from Eq. (4) that

$$\begin{aligned}
\left(\gamma + \frac{\eta k^2}{\mu_0}\right) b_r &= \imath k B_0 v_r, \\
\left(\gamma + \frac{\eta k^2}{\mu_0}\right) b_\theta - r\Omega' b_r &= \imath k B_0 v_\theta, \\
\left(\gamma + \frac{\eta k^2}{\mu_0}\right) b_z &= \imath k B_0 v_z.
\end{aligned} \tag{9}$$

Eqs. (6) to (9) seem to show that we have eight equations for seven unknowns. However, with the help of Eqs. (6) and (8), it can be easily checked that the first and third of Eqs. (9) are linearly dependent. This means that, actually, we have seven linearly independent equations for seven unknows. By choosing the third of Eqs. (9) to be a spurious equation, we may determine p in terms of v_z from the third of Eqs. (7), v_z in terms of v_r from Eq. (6), and b_z in terms of b_r from Eq. (8). As a result, we are left with the determination of the subset of four unknowns v_r, v_θ, b_r, and b_θ from the first two of both Eqs. (7) and (9) to fix the whole set of seven perturbative quantities.

5. DISPERSION RELATION

From the physical perspective, the relevant quantities are the non-trivial solutions for the whole set (6)–(9), which are obtained by requiring that the determinant of the coefficient matrix for the subset — as mentioned above, first two of both Eqs. (7) and (9) — vanishes. In this case, the (in principle, complex) time rate γ and (definitely, real) wavenumber k must satisfy the dispersion relation

$$\left(\gamma+\omega_{\mathrm{A}}^2 T\right)^2\left(\gamma+\mathrm{P_m}\omega_{\mathrm{A}}^2 T\right)^2+\left[\kappa^2\left(\gamma+\omega_{\mathrm{A}}^2 T\right)+2\omega_{\mathrm{A}}^2\left(\gamma+\mathrm{P_m}\omega_{\mathrm{A}}^2 T\right)\right]\left(\gamma+\omega_{\mathrm{A}}^2 T\right)+\left(\omega_{\mathrm{A}}^2-2q\Omega^2\right)\omega_{\mathrm{A}}^2=0, \tag{10}$$

where we have introduced the usual epicyclic frequency κ, shear parameter q, Alfvén frequency ω_{A}, magnetic Prandtl number $\mathrm{P_m}$, and decay time T of the flow perpendicular to the magnetic field, which are defined through the formulae

$$\kappa^2=2\Omega\left(2\Omega+r\Omega'\right),\qquad q\Omega=-r\Omega',\qquad \omega_{\mathrm{A}}^2=\frac{k^2B_0^2}{\mu_0\rho_0},\qquad \mathrm{P_m}=\frac{\nu/\rho_0}{\eta/\mu_0},\qquad \text{and}\qquad T=\frac{\eta\rho_0}{B_0^2}. \tag{11}$$

When dissipative effects are negligible, Eq. (10) approaches

$$\gamma^4+\left(\kappa^2+2\omega_{\mathrm{A}}^2\right)\gamma^2+\left(\omega_{\mathrm{A}}^2-2q\Omega^2\right)\omega_{\mathrm{A}}^2=0. \tag{12}$$

We see that Eq. (12) has a root $\gamma>0$ if the condition

$$\omega_{\mathrm{A}}^2<2q\Omega^2 \tag{13}$$

is satisfied, provided that $q>0$. Result (13) is nothing but the usual condition for the ideal (the plasma resistivity and fluid viscosity are negligible) MRI to occur and γ denotes its standard growth rate.

6. KEPLERIAN DISK

An important application of result (13) in astrophysics concerns the description of the instability of an accretion disk. By assuming that its differential rotation about a central object of mass M is of gravitational origin, one may easily check that

$$\Omega^2=\frac{GM}{r^3}, \tag{14}$$

where G stands for Newton's gravitational constant. Eq. (14) is nothing but Kepler's third law and the system is known as a Keplerian disk. Given Eq. (14), condition (13) shows that the wavelength $\lambda=2\pi/k$ of an unstable mode exhibits a lower limit,

$$\lambda>\frac{2\pi B_0/\Omega}{\sqrt{3\mu_0\rho_0}}, \tag{15}$$

for a Keplerian disk. However, according to Eq. (14), limit (15) is unbounded. Actually, as the central object is approached, more and more monochromatic unstable modes fit within the disk (thickness). Such a result is physically unacceptable because it states that as the radial coordinate decreases, the frequency of the unstable modes increases with no limit. Let us see how this drawback can be circumvented when both finite plasma resistivity and fluid viscosity are introduced into the problem.

7. WAVELENGTH SATURATION

For most applications, the magnetic Prandtl number, $\mathrm{P_m}$, is a very small dimensionless quantity. For instance, its order of magnitude is approximately 10^{-5} in liquid-metal laboratory dynamos or in the interiors of planets and may attain 10^{-2} at the base of the Sun's convection zone. This means that we can consider the feasible situation for which the (modulus of the complex) perturbative time rate, γ, is negligible with respect to the resistive time rate, $\omega_{\mathrm{A}}^2 T$, yet comparable to the viscous time rate, $\mathrm{P_m}\omega_{\mathrm{A}}^2 T$. In this case, Eq. (10) approaches

$$\left(\omega_{\mathrm{A}}^2 T^2\right)\left(\gamma+\mathrm{P_m}\omega_{\mathrm{A}}^2 T\right)^2+\left(2\omega_{\mathrm{A}}^2 T\right)\left(\gamma+\mathrm{P_m}\omega_{\mathrm{A}}^2 T\right)+\left[\left(1+\kappa^2 T^2\right)\omega_{\mathrm{A}}^2-2q\Omega^2\right]=0, \tag{16}$$

such that Eq. (16) has a positive root, provided that the condition

$$\left[\left(1+\mathrm{P_m}\omega_{\mathrm{A}}^2 T^2\right)^2+\left(\kappa^2 T^2\right)\right]\omega_{\mathrm{A}}^2<2q\Omega^2 \tag{17}$$

is satisfied for some also positive shear parameter. However, given the ideal MRI condition (13) and the smallness of $\mathrm{P_m}$, when resistive effects dominate, Eq. (17) approaches

$$\left(\kappa^2 T^2\right)\omega_{\mathrm{A}}^2<2q\Omega^2. \tag{18}$$

Quite interestingly, Eq. (14) shows that the lower limit for the wavelength of an unstable mode in a Keplerian disk now becomes bounded,

$$\lambda>\frac{2\pi\eta/B_0}{\sqrt{3\mu_0/\rho_0}}, \tag{19}$$

because it does not depend on the radial coordinate. This means that the frequency of the unstable modes saturates asymptotically to a maximum value, which is fully determined by the resistivity of the disk.

8. CONCLUSION

Dissipative effects have been investigated on the MRI. The basic changes with respect to the approach to the problem in the ideal limit have been: *(1)* the introduction of the resistive, η/μ_0, and viscous, ν/ρ_0, diffusivities in the induction, (4), and Navier-Stokes, (2), equations, respectively, and *(2)* the identification of the resistive, $\eta k^2/\mu_0=\omega_{\mathrm{A}}^2 T$, and viscous, $\nu k^2/\rho_0=\mathrm{P_m}\omega_{\mathrm{A}}^2 T$, time rates in the general dispersion relation, (10). As a result, when resistive effects dominate, it has been shown that the instability condition in the ideal limit, (13), is corrected by the coefficient $\kappa^2 T^2$, see Eq. (18). The proposed formulation has been then applied to the description of the instability for a Keplerian disk and it has been found that the characteristic frequency of the unstable modes saturates asymptotically to a maximum value which does not depend on the distance from the central object and is fully determined by the resistivity of the disk, see Eq. (19).

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