Hybrid TEM Wave Radiation from a Coaxial Waveguide with a Semi-infinite PEC Outer Cylinder and an Infinite Inner Cylinder Loaded with Partial Impedance

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Abstract— In this work, the radiation of a hybrid TEM mode from a coaxial waveguide with an infinitely thin semi infinite outer cylinder made of perfect electric conductor and an infinite inner cylinder lying along axial direction which has piecewise surface impedance has been analyzed through the Wiener-Hopf technique in conjunction with the mode matching method.

1. INTRODUCTION

The aim of this work is to analyze the radiation of a hybrid TEM mode from a coaxial waveguide with an infinitely thin semi infinite outer cylinder made of perfect electric conductor and an infinite inner cylinder lying along axial direction which has piecewise surface impedance as shown in Figure 1. Being a very good model for long monopole antennas or traveling wave antennas, the excitation of such structures with TE, TM and TEM modes have been analyzed extensively [1–7]. In this work, Wiener-Hopf technique in conjunction with the mode-matching method has been implemented to the radiation problem of a TM_{00} (Hybrid TEM) mode wave travelling inside the coaxial structure shown in Figure 1. By expanding the scattered field into a series of normal modes in the waveguide region and applying Fourier transform else where one gets a modified Wiener-Hopf equation. The solution involves three sets of infinite expansion coefficients satisfying three sets of infinite linear algebraic equations. The numerical results showing the effects of physical parameters have been given graphically. A time dependence $e^{-i\omega t}$ with ω being the angular frequency is assumed and suppressed throughout.

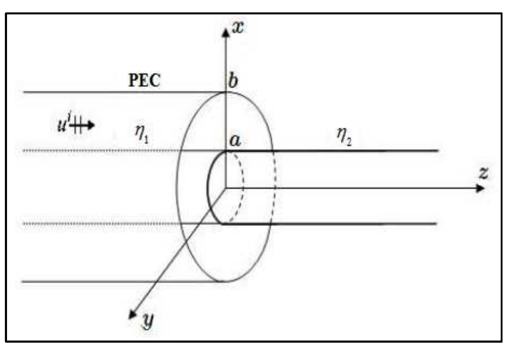


Figure 1: Geometry of the problem.

2. ANALYSIS

The waveguide analyzed here is defined by two coaxial cylinders. The inner cylinder is defined as $\rho = a$, $\phi \in (-\pi, \pi)$, $z \in (-\infty, \infty)$ while the outer cylinder is defined $\rho = b$, $\phi \in (-\pi, \pi)$, $z \in (-\infty, 0)$ where (ρ, ϕ, z) are cylindrical coordinates. The outer cylinder is infinitely thin and made of perfect electrical conductor and the inner conductor has the surface impedance $Z_1 = \eta_1 Z_0$ where $z < 0$ and $Z_2 = \eta_2 Z_0$ where $z > 0$. Here, Z_0 denotes the impedance of free space. The incident TM_{00}

(Hybrid TEM) mode has the angular frequency ω and propagates in the $+z$ direction and it is given as $H_\rho = 0$, $H_\phi = u_i$, $H_z = 0$. Total field can be written as shown below.

$$u^T(\rho, z) = \begin{cases} u_1(\rho, z), & \rho > b, \quad z \in (-\infty, \infty) \\ u_2(\rho, z), & \rho \in (a, b), \quad z > b \\ u^i(\rho, z) + u_3(\rho, z), & \rho \in (a, b), \quad z < b \end{cases} \tag{1}$$

$u_n(\rho, z)$ and $u^i(\rho, z)$ are solutions of the Helmholtz equation shown below.

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial z^2} + k^2 - \frac{1}{\rho^2}\right] u_n(\rho, z) = 0, \quad n = 1, 2, 3 \tag{2}$$

where k is the propagation constant which is assumed to have a small imaginary part corresponding of slightly lossy medium. The field terms in Equation (1) are subject to the following boundary and continuity conditions.

$$u_{1,3}(b, z) + b\frac{\partial u_{1,3}(b, z)}{\partial\rho} = 0, \quad z < 0, \tag{3}$$

$$(1 + ika\eta_{1,2})\, u_{3,2}(a, z) + a\frac{\partial u_{3,2}(a, z)}{\partial\rho} = 0, \quad z < 0, \tag{4}$$

$$u_2(b, z) = u_1(b, z), \quad z > 0, \tag{5a}$$

$$\frac{\partial u_2(b, z)}{\partial\rho} = \frac{\partial u_1(b, z)}{\partial\rho}, \quad z > 0 \tag{5b}$$

$$u^i(\rho, 0) + u_3(\rho, 0) = u_2(\rho, 0), \quad \rho \in (a, b), \tag{6a}$$

$$\frac{\partial u^i(\rho, 0)}{\partial z} + \frac{\partial u_3(\rho, 0)}{\partial z} = \frac{\partial u_2(\rho, 0)}{\partial z}, \quad \rho \in (a, b) \tag{6b}$$

In order to find a single solution for the mixed boundary value problem defined with Equations (2) and (3)–(6), the edge and radiation conditions shown below are also needed to be considered.

$$u^T(b, z) = O\left(|z|^{1/2}\right), \quad |z| \to 0, \tag{7a}$$

$$u_1(\rho, z) = O\left(\frac{e^{ik|\rho|}}{\rho}\right), \quad |\rho| \to \infty \tag{7b}$$

The Wiener-Hopf equation shown in (8) can be obtained by applying Fourier transformation is to (2) for $u_1(\rho, z)$ and $u_2(\rho, z)$ and using (3), (4), (5a), and (5b)

$$\frac{1}{N(\alpha)\, M(\alpha)} P^+(\alpha) - \frac{F^-(b, \alpha)}{ik\eta_2} = -b\sum_{m=0}^{\infty} \frac{[f_m - i\alpha g_m]\,[J_m Y_1(K_m b) - Y_m J_1(K_m b)]}{\alpha^2 - \alpha_m^2} \tag{8}$$

$$M(\alpha) = K(\alpha)\,[J(\alpha)\, Y_0[K(\alpha)\, b] - Y(\alpha)\, J_0[K(\alpha)\, b]], \tag{9a}$$

$$N(\alpha) = \frac{H_0^1(Kb)}{H_1^1(Kb)\, M_0(\alpha) + H_0^1(Kb)\, M_1(\alpha)} \tag{9b}$$

$$J(\alpha) = ik\eta_2 J_1[K(\alpha)\, a] + K(\alpha)\, J_0[K(\alpha)\, a], \tag{10a}$$

$$Y(\alpha) = ik\eta_2 Y_1[K(\alpha)\, a] + K(\alpha)\, Y_0[K(\alpha)\, a] \tag{10b}$$

$$M_j(\alpha) = J(\alpha)\, Y_j(Kb) - Y(\alpha)\, J_j(Kb), \quad j = 0, 1 \tag{11}$$

$$P^+(\alpha) = G^+(b, \alpha) + b\dot{G}^+(b, \alpha) = F^+(b, \alpha) + b\dot{F}^+(b, \alpha), \tag{12a}$$

$$P^+(\alpha_m) = -\frac{\pi}{2} K_m b\,[f_m - i\alpha_m g_m]\,\Delta_m \tag{12b}$$

Here, $(\dot{})$ denotes derivation in terms of ρ.

$$F^{\pm}(\rho,\alpha)=\pm\int_{0}^{\pm\infty}u_1(\rho,z)\,e^{i\alpha z}dz, \tag{13a}$$

$$G^{+}(\rho,\alpha)=\int_{0}^{\infty}u_2(\rho,z)\,e^{i\alpha z}dz \tag{13b}$$

$$\begin{bmatrix}f_m\\ g_m\end{bmatrix}=\frac{1}{\Delta_m}\int_{a}^{b}\begin{bmatrix}f(\rho)\\ g(\rho)\end{bmatrix}\left[J_0(K_mb)\,Y_1(K_m\rho)-J_1(K_m\rho)\,Y_0(K_mb)\right]\rho d\rho \tag{14}$$

$$f(\rho)=\frac{\partial}{\partial z}u_2(\rho,0), \tag{15a}$$
$$g(\rho)=u_2(\rho,0) \tag{15b}$$

$$\begin{bmatrix}f(\rho)\\ g(\rho)\end{bmatrix}=\sum\nolimits_{m=0}^{\infty}\begin{bmatrix}f_m\\ g_m\end{bmatrix}\left[J_0(K_mb)\,Y_1(K_m\rho)-J_1(K_m\rho)\,Y_0(K_mb)\right] \tag{16}$$

$$\Delta_m=\frac{b^2}{2}\left(\frac{2}{\pi K_mb}\right)^2-\frac{a^2}{2}\left[1+\frac{ik\eta_2}{K_m^2a}(2+ik\eta_2a)\right]\tilde{L}_1^2, \tag{17a}$$
$$\tilde{L}_1=J_0(K_mb)\,Y_1(K_ma)-J_1(K_ma)\,Y_0(K_mb) \tag{17b}$$

$K(\alpha)=\sqrt{k^2-\alpha^2}$ is the square root function defined on the complex α-plane and $K_m=K(\alpha_m)$. Where,

$$M(\alpha_m)=0, Im(\alpha_m)>Im(-k), \quad m=0,1,2,\ldots \tag{18}$$

3. THE SOLUTION OF THE WIENER-HOPF EQUATION

By applying the Wiener-Hopf procedure to (8), we obtain (19).

$$\frac{P^{+}(\alpha)}{V^{+}(\alpha)}=b\sum\nolimits_{m=0}^{\infty}\frac{[f_m-i\alpha_mg_m]\left[J_mY_1(K_mb)-Y_mJ_1(K_mb)\right]N^{+}(\alpha_m)\,M^{+}(\alpha_m)}{2\alpha_m(\alpha+\alpha_m)}, \tag{19a}$$
$$V(\alpha)=M(\alpha)\,N(\alpha)=V^{+}(\alpha)\,V^{-}(\alpha) \tag{19b}$$

$V^{+}(\alpha)$ and $V^{-}(\alpha)$ which are regular and don't have zeros in the regions $\mathrm{Im}(\alpha)<\mathrm{Im}(k)$ and $\mathrm{Im}(\alpha)>\mathrm{Im}(k)$ respectively. $V^{+}(\alpha)$ is obtained as below using the procedures explained in [9].

$$V^{+}(\alpha)=\sqrt{V(0)}e^{\left\{\frac{i\alpha(b-a)}{\pi}\left[1-C-\ln\left(\frac{2\pi}{k(b-a)}\right)+i\frac{\pi}{2}\right]+\frac{K(\alpha)(b-a)}{\pi}\ln\left(\frac{\alpha+iK(\alpha)}{k}\right)-\frac{1}{2}i(b-a)K(\alpha)+q(\alpha)\right\}}$$
$$\prod\nolimits_{l=1}^{L}\left(\frac{\beta_l-\alpha}{\beta_l+\alpha}\right)^{1/2}\prod\nolimits_{m=0}^{\infty}\left(1+\frac{\alpha}{\alpha_m}\right)e^{\left(\frac{i\alpha(b-a)}{m\pi}\right)} \tag{20}$$

$$q(\alpha)=\frac{1}{2}P\int_{0}^{\infty}K_w(w)\ln\left[\frac{\sqrt{k^2-w^2}+\alpha}{\sqrt{k^2-w^2}-\alpha}\right]dw, \tag{21a}$$

$$K_w(w)=\frac{(b-a)}{\pi}+\frac{1}{2\pi i}\left[B_w(w)+B_w\left(we^{i\pi}\right)\right] \tag{21b}$$

$$B_w(w)=\frac{bH_1^{(1)}(wb)}{H_0^{(1)}(wb)}+\frac{\left[H_1^{(1)}(wb)-H_0^{(1)}(wb)\right]\begin{bmatrix}T_{00}(w) & T_{01}(w)\\ T_{10}(w) & T_{11}(w)\end{bmatrix}\begin{bmatrix}ik\eta a\\ -\left(\frac{i2k\eta}{w}+a\right)\end{bmatrix}}{\left[H_1^{(1)}(wb)-H_0^{(1)}(wb)\right]\begin{bmatrix}T_{00}(w) & T_{01}(w)\\ T_{10}(w) & T_{11}(w)\end{bmatrix}\begin{bmatrix}w\\ ik\eta\end{bmatrix}} \tag{22}$$

$$T_{ij}(w)=Y_i(wb)\,J_j(wa)-Y_j(wa)\,J_i(wb), \quad i,j=0,1 \tag{23}$$

C in (20) is Euler constant which is $C = 0.57721\ldots$. $\pm\beta_l$ are the roots of $N(\alpha)$. P in (21a) signifies that Cauchy principle value of the integral has been considered.

Let the axially symmetric TM_{00} mode incident field be represented by [1]

$$u^i = \left[J_1\left(\xi_0\rho\right) - \frac{J_0\left(\xi_0 b\right)}{Y_0\left(\xi_0 b\right)} Y_1\left(\xi_0\rho\right)\right] e^{i\beta_0 z}, \tag{24}$$

where β_0 is the propagation constant for dominant mode related to ξ_0 by the equation $\beta_n = \sqrt{k^2 - \xi_n^2}$, $n = 0, 1, 2, \ldots$ with $n = 0$. ξ_n is determined by the characteristic equation

$$\frac{ik\eta_1\left[J_1\left(\xi_n a\right) Y_0\left(\xi b\right) - J_0\left(\xi_n b\right) Y_1\left(\xi_n a\right)\right] + \xi\left[J_0\left(\xi_n a\right) Y_0\left(\xi_n b\right) - J_0\left(\xi_n b\right) Y_0\left(\xi_n a\right)\right]}{Y_0\left(\xi_n b\right)} = 0 \tag{25}$$

Size of the waveguide is chosen such that only the one real root of the equation will be allowed. This signifies the dominant mode. The important point to be noted is that only the inductive guiding surface ($\eta_1 = iX_1$) may support the TM_{00} mode. In the waveguide region the scattered field $u_3(\rho, z)$ can be expanded into a series of normal modes as

$$u_3\left(\rho, z\right) = c_0\left[J_1\left(\xi_0\rho\right) - \frac{J_0\left(\xi_0 b\right)}{Y_0\left(\xi_0 b\right)} Y_1\left(\xi_0\rho\right)\right] e^{-i\beta_0 z} + \sum_{n=1}^{\infty} c_n\left[J_1\left(\xi_n\rho\right) - \frac{J_0\left(\xi_n b\right)}{Y_0\left(\xi_n b\right)} Y_1\left(\xi_n\rho\right)\right] e^{-i\beta_n z} \tag{26}$$

The first term of the right side of (26) describes the reflected field for the dominant mode.

The unknown coefficients f_m, g_m and c_n appearing in (12b) and (26), respectively are to be determined trough the following matching conditions derived from continuity relations (6a), (6b)

$$\Delta_m g_m = \left(c_0 + 1\right) I_0^{(1)} + \sum_{n=1}^{\infty} c_n I_n^{(1)}, \tag{27a}$$

$$\Delta_m f_m = i\beta_0\left(1 - c_0\right) I_m^{(1)} - \sum_{n=1}^{\infty} i\beta_n c_n I_n^{(1)} \tag{27b}$$

$$-\frac{\pi K_m b}{2}\left[f_m - i\alpha_m g_m\right]\Delta_m = N^+\left(\alpha_m\right) M^+\left(\alpha_m\right) b \sum_{n=0}^{\infty} \frac{\left[f_n - i\alpha_n g_n\right] N^+\left(\alpha_n\right) M^+\left(\alpha_n\right)}{2\alpha_n\left(\alpha_m + \alpha_n\right)} I_n^{(2)},$$
$$m = 0, 1, 2, \ldots \tag{28}$$

with

$$I_n^{(1)} = \int_a^b \left[J_1\left(\xi_n\rho\right) - \frac{J_0\left(\xi_n b\right)}{Y_0\left(\xi_n b\right)} Y_1\left(\xi_n\rho\right)\right]\left[J_0\left(K_m b\right) Y_1\left(K_m\rho\right) - J_1\left(K_m\rho\right) Y_0\left(K_m b\right)\right]\rho d\rho \tag{29}$$

$$I_n^{(2)} = J_n Y_1\left(K_n b\right) - Y_n J_1\left(K_n b\right), \tag{30}$$

where $Y_n = Y(\alpha_n)$, $J_n = J(\alpha_n)$. (27a), (27b) (28)–(30) form an infinite system of linear algebraic equations which is used to calculate the unknown coefficients.

4. SCATTERED FIELDS AND NUMERICAL RESULTS

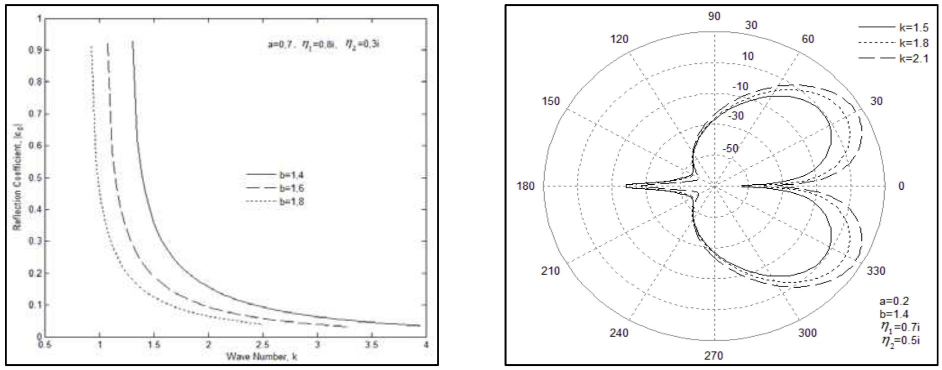
The radiated field $u_1(\rho, z)$ can be obtained by evaluating the following integral

$$u_1\left(\rho, z\right) = \int_L \frac{P^+\left(\alpha\right)}{bK\left(\alpha\right) H_0^{(1)}\left[K\left(\alpha\right) b\right]} H_1^{(1)}\left[K\left(\alpha\right)\rho\right] e^{-i\alpha z} d\alpha \tag{31}$$

The evaluation of the above integral by using steepest descent path method gives rise to the far field asymptotic expression of $u_1(\rho, z)$ as follows:

$$u_1\left(r, \theta\right) = H\left(\theta\right)\frac{e^{ikr}}{kr}, \quad H\left(\theta\right) = -\frac{1}{\pi b\sin\theta}\frac{F^+\left(-k\cos\theta\right)}{H_0^{(1)}\left[kb\sin\theta\right]} \tag{32}$$

Here r and θ denote spherical coordinates defined by $\rho = r\sin\theta$, $z = r\cos\theta$.



(a) (b)

Figure 2: (a) Variation of the amplitude of the reflection coefficient $|c_0|$ against$k$ for different values of outer radius b. (b) Variation of the $20\log|u_1|$ versus observation angle for different values of k.

The coefficient c_0 corresponds to the amplitude of the reflected field is obtained by numerical calculation of the infinite system of linear algebraic Equations (27)–(30). Figure 2(a) shows the variation of the amplitude of the reflection coefficient $|c_0|$, against the wave number for three different values of b while $a = 0.7$, $\eta_1 = 0.8i$, $\eta_2 = 0.3i$. It is seen that the amplitude of the reflection coefficient decreases rapidly with increasing values of the frequency. Figure 2(b) shows the variation of the amplitude of the radiated field, $20\log|u_1|$ against observation angle for three different values of wave number k while $a = 0.2$, $b = 1.4$, $\eta_1 = 0.7$, $\eta_2 = 0.5i$. It is observed that the amplitude of the radiated field increases especially observation angle θ between $(0°, 60°)$ degrees with increasing values of the frequency.

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