Permittivity Reconstruction of a Diaphragm in a Rectangular Waveguide: Unique Solvability of Benchmark Inverse Problems

Y. Shestopalov[2], **Y. Smirnov**[1], **and E. Derevyanchuk**[1]

[1]Penza State University, Russia
[2]University of Gävle, Sweden

Abstract— This work is devoted to the analysis of inverse problems of permittivity reconstruction of diaphragms loaded in a rectangular waveguide including justification of the solution to a benchmark inverse problem of reconstructing permittivity of a one-sectional diaphragm. We perform comparison of theoretical and numerical results based on the measurement data to validate the efficiency of the proposed technique. The obtained solutions can be implemented in practical measurements for investigation of new artificial materials and media and can be applied in optics, nanotechnology, and design of microwave devices.

1. INTRODUCTION

Determination of electromagnetic parameters of dielectric bodies is an urgent problem for artificial materials: nanomaterials, metamaterials, and nanocomposite materials. As a rule, electromagnetic parameters of such kind of materials cannot be measured. This work is a continuation of the series of papers [1–3] devoted to the investigation of permittivity reconstruction of layered materials in the form of diaphragms (sections) in a single-mode waveguide of rectangular cross section from the transmission coefficient measured at different frequencies. The aim is the development of the mathematical technique for the analysis of the inverse problem for the case of a one- and multi-sectional diaphragms in a rectangular waveguide. We suppose that the diaphragm is made of material with real electromagnetic parameters. For this case, we obtain analytical solution of the inverse problem and the conditions providing unique solvability for the benchmark case of a one-sectional diaphragm. Based on the analytical solution we study more complicated mathematical models and present results of numerical simulation.

2. INVERSE PROBLEM

Assume that a waveguide $P = \{x : 0 < x_1 < a, 0 < x_2 < b, -\infty < x_3 < \infty\}$ with the perfectly conducting boundary surface ∂P is given in Cartesian coordinate system. A three-dimensional body Q ($Q \subset P$)

$$Q = \{x : 0 < x_1 < a, 0 < x_2 < b, 0 < x_3 < l\}$$

is placed in the waveguide; the body has the form of a diaphragm (an insert), namely, a parallelepiped separated into n sections adjacent to the waveguide walls. Domain $P\backslash\bar{Q}$ is filled with an isotropic and homogeneous layered medium having constant permeability ($\mu_0 > 0$) in whole waveguide P, the sections of the diaphragm

$$\begin{aligned} Q_0 &= \{x : 0 < x_1 < a, 0 < x_2 < b, -\infty < x_3 < 0\} \\ Q_j &= \{x : 0 < x_1 < a, 0 < x_2 < b, l_{j-1} < x_3 < l_j\}, j = 1, \dots, n \\ Q_{n+1} &= \{x : 0 < x_1 < a, 0 < x_2 < b, l < x_3 < +\infty\} \end{aligned}$$

are filled each with a medium having constant permittivity $\varepsilon_j > 0$; $l_0 := 0$, $l_n := l$.

The electromagnetic field inside and outside of the object in the waveguide is governed by Maxwells' equations:

$$\begin{aligned} \mathrm{rot}\mathbf{H} &= -i\omega\varepsilon\mathbf{E} \\ \mathrm{rot}\mathbf{E} &= i\omega\mu_0\mathbf{H}, \end{aligned} \tag{1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the vectors of the electric and magnetic field intensity and ω is the circular frequency. Assume that $\pi/a < k_0 < \pi/b$, where k_0 is the wavenumber, $k_0^2 = \omega^2\varepsilon_0\mu_0$ [5]. In this case, only one wave H_{10} propagates in the waveguide without attenuation (we have a single-mode waveguide [5]).

The incident electrical field is

$$\mathbf{E}^0 = \mathbf{e}_2 A \sin\left(\frac{\pi x_1}{a}\right) e^{-i\gamma_0 x_3} \tag{2}$$

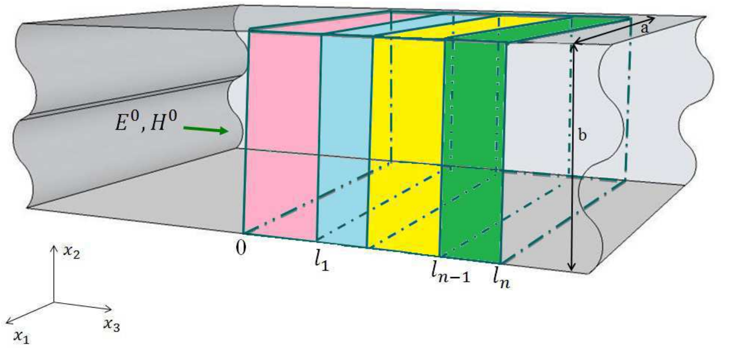

Figure 1: Multilayered diaphragms in a waveguide.

with a known A and $\gamma_0 = \sqrt{k_0^2 - \pi^2/a^2}$.

Solving the forward problem for Maxwell's equations we obtain explicit expressions for the field inside every section of diaphragm Q and outside the diaphragm:

$$E_{(0)} = \sin\left(\frac{\pi x_1}{a}\right)(Ae^{-i\gamma_0 x_3} + Be^{i\gamma_0 x_3}), \quad x \in Q_0, \tag{3}$$

$$\begin{aligned} E_{(j)} &= \sin\left(\frac{\pi x_1}{a}\right)(C_j e^{-i\gamma_j x_3} + D_j e^{i\gamma_j x_3}), \\ j &= 1, \ldots, n+1; \qquad D_{n+1} = 0, \quad x \in Q_j, \end{aligned} \tag{4}$$

where $\gamma_j = \sqrt{k_j^2 - \pi^2/a^2}$ and $k_j^2 = \omega^2 \varepsilon_j \mu_0$, $\gamma_{n+1} = \gamma_0$.

From the conditions on the boundary surfaces $L := \{x_3 = 0, x_3 = l_1, \ldots, x_3 = l_n\}$ of the diaphragm sections

$$[\mathbf{E}]|_L = 0; [\mathbf{H}]|_L = 0, \tag{5}$$

where square brackets $[\cdot]$ denote function jump via boundary surfaces, applied to (3) and (4) we obtain using conditions (5) a system of equations for the unknown coefficients

$$\begin{cases} A + B = C_1 + D_1 \\ \gamma_0 (B - A) = \gamma_1 (D_1 - C_1) \\ C_j e^{-i\gamma_j l_j} + D_j e^{i\gamma_j l_j} = C_{j+1} e^{-i\gamma_{j+1} l_j} + D_{j+1} e^{i\gamma_{j+1} l_j} \\ \gamma_j (D_j e^{i\gamma_j l_j} - C_j e^{-i\gamma_j l_j}) = \gamma_{j+1} (D_{j+1} e^{i\gamma_{j+1} l_j} - C_{j+1} e^{-i\gamma_{j+1} l_j}), \quad j = 1, \ldots, n. \end{cases} \tag{6}$$

where $C_{n+1} = F$, $D_{n+1} = 0$. In system (6) coefficients A, B, C_j, D_j are supposed to be complex and ε_j, $(j = 1, \ldots, n)$ real.

Formulate the inverse problem for a multisectional diaphragm.

Inverse problem P: *find (real) permittivity ε_j of each section from the known amplitude A of the incident wave and amplitude F of the transmitted wave at different frequencies.*

In the case of an n-sectional diaphragm, we have [1–3] a system of n equations with n unknown permittivities

$$\mathrm{Re}(A(\omega_j)) = \mathrm{Re}\left(\frac{1}{2\prod\limits_{j=0}^{n} \gamma_j} (\gamma_n p_{n+1} + \gamma_0 q_{n+1}) F(\omega_j) e^{-i\gamma_0 l_n}\right), \quad j = 1, \ldots, n, \tag{7}$$

where p and q are obtained from recurrent formulas

$$p_{j+1} = \gamma_{j-1} p_j \cos\alpha_j + \gamma_j q_j i \sin\alpha_j; \quad p_1 := 1,$$

$$q_{j+1} = \gamma_{j-1} p_j i \sin\alpha_j + \gamma_j q_j \cos\alpha_j; \quad q_1 := 1.$$

Here $\alpha_j = \gamma_j (l_j - l_{j-1})$, $j = 1, \ldots, n$. Solving system (7) numerically we find permittivities ε_j $(j = 1, \ldots, n)$.

3. EXISTENCE AND UNIQUENESS

Consider theoretical results including the proofs of uniqueness and solvability of the inverse problems. For a one-sectional diaphragm from (7) we obtain the equation

$$\frac{F}{A} = \frac{e^{i\gamma_0 l_1}}{g(\tau)}, \tag{8}$$

$$g(\tau) = \cos(\tau) + i\left(\frac{\tau}{2\gamma_0 l_1} + \frac{\gamma_0 l_1}{2\tau}\right)\sin(\tau), \tag{9}$$

$$\tau = \gamma_1 l_1 = l_1\sqrt{\omega^2\varepsilon_1\mu_0 - \frac{\pi^2}{a^2}}. \tag{10}$$

If $\varepsilon_1 > \varepsilon_0 > 0$ then

$$\tau = \gamma_1 l_1 = l_1\sqrt{k_1^2 - \frac{\pi^2}{a^2}} > 0. \tag{11}$$

Introduce a complex variable $z = x + iy = Ae^{i\gamma_0 l_1}/F$ and separate real and imaginary parts in (8):

$$\begin{cases} x = \cos\tau \\ y = h(\tau)\sin\tau, \end{cases} \tag{12}$$

where $h(\tau) = \frac{\tau}{2\gamma_0 l_1} + \frac{\gamma_0 l_1}{2\tau}$. Equation (8) is equivalent to the system

$$\begin{cases} \cos\tau = \rho, \\ h(\tau)\sin\tau = \zeta, \end{cases} \tag{13}$$

where

$$\rho = \mathrm{Re}\left(Ae^{i\gamma_0 l_1}/F\right), \quad \zeta = \mathrm{Im}\left(Ae^{i\gamma_0 l_1}/F\right) \tag{14}$$

The following theorems justify unique solvability of the inverse problem under study.

Theorem 1 *Suppose that $|\rho| < 1$ and $\rho^2 + \zeta^2 \geq 1$. Then inverse problem P has a unique solution*:

$$\varepsilon_1 = \frac{1}{\omega^2\mu_0\varepsilon_0}\left(\left(\frac{\pi}{a}\right)^2 + \left(\frac{\tau}{l_1}\right)^2\right), \tag{15}$$

where

$$\tau = \tau_1 = \gamma_0 l_1\left(\frac{|\zeta| + \sqrt{\rho^2 + \zeta^2 - 1}}{\sqrt{1-\rho^2}}\right), \tag{16}$$

if the conditions $\tau_1/\gamma_0 l_1 > 1$, $\varepsilon_1 > \varepsilon_0$, $\cos\tau_1 = \rho$, and sign $(\zeta) =$ sign $(\sin\tau_1)$ are satisfied.

If $\tau_2/\gamma_0 l_1 < 1$, $\pi^2/a^2\omega^2\mu_0 < \varepsilon_1 < \varepsilon_0$, $\cos\tau_2 = \rho$, and sign $(\zeta) =$ sign $(\sin\tau_2)$, then the inverse problem has only one solution expressed in the form

$$\tau = \tau_2 = \gamma_0 l_1\left(\frac{\sqrt{1-\rho^2}}{|\zeta| + \sqrt{\rho^2 + \zeta^2 - 1}}\right), \tag{17}$$

Otherwise, inverse problem P has no solution.

Remark 1. If $\rho = \pm 1$ then ζ must be equal to zero and, respectively, $\tau = 2\pi n$ or $\tau = \pi + 2\pi n$ $(n \in Z)$. In these cases uniqueness is violated and inverse problem P has several solutions.

Theorem 2 *If*

$$F = \pm Ae^{i\gamma_0 l_1}, \tag{18}$$

or

$$F = \frac{Ae^{i\gamma_0 l_1}}{g(\tau_1^*)}, \tag{19}$$

where

$$\tau_1^* = \frac{C^2}{\pi n + \sqrt{\pi^2 n^2 + C^2}}, \quad C = \gamma_0 l_1, \quad n \geq 0, \tag{20}$$

then inverse problem P has no solution in the case of a one-sectional diaphragm. If these conditions are not satisfied, then inverse problem P has a unique solution.

Proof. From (7), we have

$$g(\tau) = \frac{Ae^{i\gamma_0 l_1}}{F}$$

Suppose that $\tau \geq 0$, τ_1 and τ_2 ($\tau_2 \geq \tau_1$) exist, and $g(\tau_1) = \frac{Ae^{i\gamma_0 l_1}}{F}$ and $g(\tau_2) = \frac{Ae^{i\gamma_0 l_1}}{F}$. Then the relation $g(\tau_1) - g(\tau_2) = 0$ holds. From (8) we have that the last equality yields a system

$$\begin{cases} \cos\tau_1 - \cos\tau_2 = 0, \\ \left(\frac{\tau_1}{C} + \frac{C}{\tau_1}\right)\sin\tau_1 - \left(\frac{\tau_2}{C} + \frac{C}{\tau_2}\right)\sin\tau_2 = 0. \end{cases} \tag{21}$$

From the first equation of system (21) we obtain

$$\cos\tau_1 - \cos\tau_2 = -2\sin\frac{\tau_1 + \tau_2}{2}\sin\frac{\tau_2 - \tau_1}{2} = 0.$$

Then $\sin\frac{\tau_1+\tau_2}{2} = 0$, or $\sin\frac{\tau_2-\tau_1}{2} = 0$. In the first case, we obtain

$$\begin{cases} \tau_1 = \pi m, \ m \geq 0, \\ \tau_2 = -\pi m + 2\pi k, \end{cases}$$

and $g(\tau_1) = g(\tau_2) = (-1)^l$ which yields the proof of the first part of the theorem (relation (18)).

In the second case when $\sin\frac{\tau_2-\tau_1}{2} = 0$, we obtain

$$\begin{cases} \tau_1^* = \frac{C^2}{\pi n + \sqrt{\pi^2 n^2 + C^2}}, \\ \tau_2 = \tau_1^* + 2\pi k, n \geq 0, \end{cases}$$

and the second part of the theorem, (19) and (20), is proved.

Remark 2. Theorem 2 yields existence and uniqueness of the solution to inverse problem P when a one-sectional diaphragm is filled with a medium having real permittivity.

Statement. *In the case of a one-sectional diaphragm filled with a medium having real permittivity inverse problem P has no unique solution if only the relative amplitude $|F/A|$ is known.*

The presence of several inresection points on the graph in Fig. 2 clearly illustrates the validity of this statement.

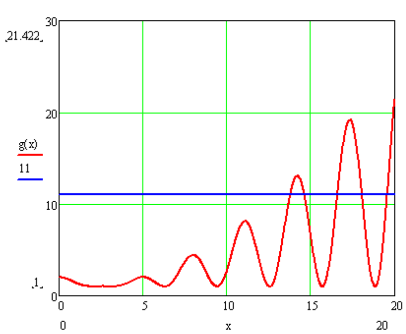

Figure 2: Graphs of functions $|g(z)|^2$. $|g(z)|^2 = \cos^2 z + \frac{1}{4}\left(\frac{z}{C} + \frac{C}{z}\right)^2 \sin^2 z$.

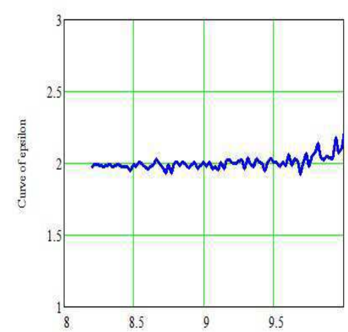

Figure 3: Real permittivity calculated from experimentally obtained values of the transmission coefficient using Equations (15) and (16), the exact value is $\varepsilon(\omega) = 2.04$, frequency range $f \in (8.2, 10)$ GHz.

4. NUMERICAL

In this section we present an example of numerical solutions to inverse problem P for a one-sectional diaphragm. Parameters of the diaphragm are $a = 2.274$ cm, $b = 1.004$ cm, and $l_1 = 1.004$ cm, and the excitation frequency $f \in (8.2, 10)$ GHz.

The permittivity was calculated on a grid of 120 (and more) frequency points taken from the indicated range using experimentally obtained values [6] of the transmission coefficient, which explains the presence of small oscillations on the graph. We see that the average permittivity value coincides well (with accuracy less than 0.5 %) with the mean value (expectation) of the calculated data. Note that the desired permittivity value (of a homogeneous one-sectional diaphragm) can be obtained, using the developed solution to inverse problem P, from *one* (complex) value of the transmission coefficient.

5. CONCLUSION

In this work we completed theoretical justification of the solution to a benchmark inverse problem of reconstructing permittivity of a one-sectional diaphragm, proving the existence and uniqueness in the case of real permittivity. We performed comparison of results of theoretical solution to the inverse problems with the numerical data employing experimentally obtained values of the transmission coefficient, validating the efficiency of the proposed technique.

The obtained solutions can be implemented in practical measurements when new artificial materials and media are investigated by commonly used waveguide devices and analyzers.

ACKNOWLEDGMENT

This work is partially supported by the grant of the Ministry of Education and Science of Russian Federation 2.11.02.2014/K and Scholarship of the President of Russia 1311.2015.5.

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