Electromagnetic Scattering-matrix Theories Based on Plane Waves and Complex-source Beams

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Abstract— Two scattering-matrix theories for time-harmonic fields in three dimensions are presented: (i) a plane-wave theory with a directional spectrum that is obtained through a complex-source point substitution procedure, and (ii) a complex-source beam theory based on a beam expansion of spherical multipole fields. Scattering matrices for plane-wave expansions, which determine the plane-wave spectrum of the scattered field of an object due to an incoming plane wave, are readily available. The analogous scattering matrices based on complex-source beams will be derived from Waterman's T matrices. These scattering matrices determine the beam weights for the scattered field in terms of the output of elementary beam receivers, which sample the incident field at complex points in space. The two scattering-matrix formulations will also be compared with Kerns plane-wave theory.

1. INTRODUCTION

Plane waves and complex-source beams constitute complete sets of basis functions for electromagnetic fields in homogeneous source-free regions. From the completeness of these basis functions, we derive exact scattering-matrix theories, which can lead to efficient computation schemes for electromagnetic field transformations in both near and far-field regions. The plane-wave basis functions have sources of infinite extent whereas the complex-source beams have sources of finite extent. Hence, the two types of expansions are distinctly different in many ways.

Figure 1 shows the electromagnetic scattering problem under consideration. The primary source generates the electric field $\mathbf{E}_p(\mathbf{r})$ that interacts with the scatterer to produce the scattered electric field $\mathbf{E}_s(\mathbf{r})$. Multiple interactions between the primary source and the scatterer are neglected. The minimum sphere of radius R_s centered at the origin encloses the scatterer, and the primary source region and the observation region are assumed to be outside this sphere. The global (x, y, z) coordinate system has origin near the scatterer. Further, $\mathbf{r}_p$ is a point in the source region that generates the primary field, and $\mathbf{r}_o$ is a point in the observation region.

The standard rectangular coordinates are denoted by (x, y, z) with unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, so that a general point in space can be expressed as $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. The spherical coordinates (r, θ, ϕ) are related to the rectangular coordinates through $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$. The three spherical unit vectors are $\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$, $\hat{\boldsymbol{\theta}} =$

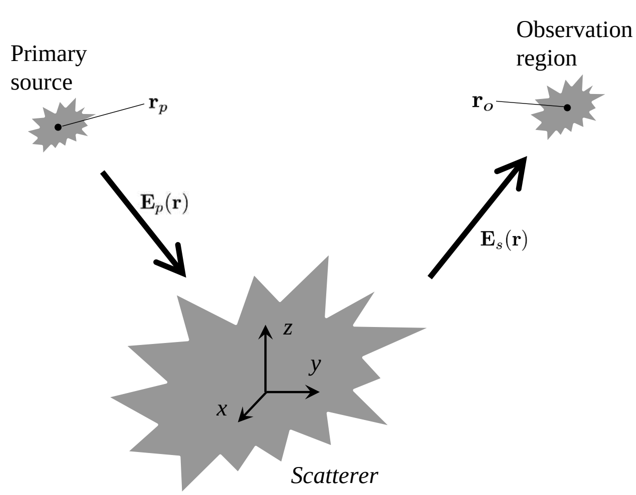

Figure 1: The scattering configuration. The field $\mathbf{E}_p(\mathbf{r})$ of the primary source encounters the scatterer. The scattered field $\mathbf{E}_s(\mathbf{r})$ is observed in the observation region. $\mathbf{r}_p$ is a point in the source region and $\mathbf{r}_o$ is a point in the observation region.

$\cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$, and $\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$. Throughout, $e^{-i\omega t}$ time dependence with $\omega > 0$ is assumed and suppressed. The wave number is $k = 2\pi/\lambda = \omega/c$, with c being the wave speed and λ the wave length.

2. EXPANSION BASED ON COMPLEX-SOURCE BEAMS

The complex-source beam of the present paper (also referred to as a Gaussian beam) was first obtained by Deschamps [1] by inserting a complex source point $\mathbf{r}' = ia\hat{\mathbf{r}}'$ with $a > 0$ into the scalar free-space Green's function to get

$$G\left(\mathbf{r}, ia\hat{\mathbf{r}}'\right) = \frac{e^{ik\sqrt{(\mathbf{r}-ia\hat{\mathbf{r}}')^2}}}{4\pi\sqrt{\left(\mathbf{r}-ia\hat{\mathbf{r}}'\right)^2}} \tag{1}$$

where the distance $\sqrt{\left(\mathbf{r}-ia\hat{\mathbf{r}}'\right)^2}$ is complex with the square root defined to have a non-negative real part and its branch cut placed along the negative real axis [2]. For fixed $\hat{\mathbf{r}}'$ and varying $\mathbf{r}$ the branch cut manifests itself in real 3D space by a branch-cut disk of radius a, centered at the origin with normal $\hat{\mathbf{r}}'$. Hence, the field $G(\mathbf{r}, ia\hat{\mathbf{r}}')$ satisfies the homogeneous Helmholtz equation everywhere except on this branch-cut disk where the sources reside.

The far-field formula (valid for large r)

$$G\left(\mathbf{r}, ia\hat{\mathbf{r}}'\right) \sim \frac{e^{ikr}}{4\pi r}\, e^{ka\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'} \tag{2}$$

shows that for $k > 0$ and $a > 0$ the beam radiates most strongly in the direction $\hat{\mathbf{r}}'$ and most weakly in the direction $-\hat{\mathbf{r}}'$. The name "Gaussian beam" is sometimes used because $G(\mathbf{r}, ia\hat{\mathbf{r}}')$ exhibits Gaussian behavior near the direction $\hat{\mathbf{r}}'$: let $\cos\Theta = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'$ in (2) to get $G(\mathbf{r}, ia\hat{\mathbf{r}}') \sim [e^{ikr}/(4\pi r)]\, e^{ka(1-\Theta^2/2)}$ when Θ is small.

2.1. The Scattered Field

According to [3, Eq. (38)], the scattered field can be expressed in terms of complex-source beams with beam parameter a_s as ($\int d\Omega'$ denotes the integral over the unit sphere with respect to $\hat{\mathbf{r}}'$)

$$\mathbf{E}_s(\mathbf{r}) = \int\! d\Omega' G(\mathbf{r}, ia_s\hat{\mathbf{r}}')\left[\mathbf{F}_M(\mathbf{r}, ia_s\hat{\mathbf{r}}')W_A\left(\theta',\phi'\right) + \mathbf{F}_N(\mathbf{r}, ia_s\hat{\mathbf{r}}')W_B(\theta',\phi')\right] \tag{3}$$

with

$$\mathbf{F}_M(\mathbf{r}, \mathbf{r}') = \left(\frac{1}{R^2} - \frac{ik}{R}\right)\mathbf{r}'\times\mathbf{r}, \quad R = \sqrt{(\mathbf{r}-\mathbf{r}')^2}, \tag{4}$$

$$\mathbf{F}_N(\mathbf{r}, \mathbf{r}') = \frac{2}{k}\left(\frac{1}{R^2} - \frac{ik}{R}\right)\mathbf{r}' + \left(\frac{3}{R^2} - \frac{3ik}{R} - k^2\right)\frac{\mathbf{r}\mathbf{R}\cdot\mathbf{r}' - \mathbf{r}'\mathbf{R}\cdot\mathbf{r}}{kR^2}, \quad \mathbf{R} = \mathbf{r} - \mathbf{r}' \tag{5}$$

and

$$W_A(\theta,\phi) = \sum_{\ell=1}^{L}\sum_{m=-\ell}^{\ell}\frac{A_{\ell m}\, Y_{\ell m}(\theta,\phi)}{\sqrt{\ell(\ell+1)}\, ik\, j_\ell(ika_s)}, \quad W_B\left(\theta,\phi\right) = \sum_{\ell=1}^{L}\sum_{m=-\ell}^{\ell}\frac{B_{\ell m}\, Y_{\ell m}(\theta,\phi)}{\sqrt{\ell(\ell+1)}\, ik\, j_\ell(ika_s)} \tag{6}$$

where $Y_{\ell m}(\theta,\phi)$ is the spherical harmonic function and $j_\ell(Z)$ is the spherical Bessel function. Also, $A_{\ell m}$ and $B_{\ell m}$ are the spherical expansion coefficients of the scattered field to be determined below. Moreover, for non-resonant scatterers the truncation number is

$$L = \mathrm{int}\left(kR_s + \gamma(kR_s)^{1/3}\right), \quad \gamma = \frac{(-3\ln\mathcal{E})^{2/3}}{2} \tag{7}$$

with $\mathcal{E}$ being the desired relative accuracy, and "ln" and "int" denoting the natural logarithm and integer part, respectively.

2.2. The Primary Field

The spherical expansion coefficients for the standing spherical-wave representation of the primary field can be expressed in terms of the output of elementary complex-point receivers as [3, Eqs. (52)–(53)]

$$C_{\ell m} = \frac{1}{j_\ell(ika_p)} \int d\Omega \mathbf{E}_p(ia_p\hat{\mathbf{r}}) \cdot \mathbf{M}^*_{\ell m}(\theta,\phi), D_{\ell m} = \frac{1}{v^{(1)}_\ell(ika_p)} \int d\Omega \mathbf{E}_p(ia_p\hat{\mathbf{r}}) \cdot \mathbf{N}^*_{\ell m}(\theta,\phi) \tag{8}$$

where $a_p > 0$ is a beam parameter, $*$ denotes complex conjugation, $v^{(1)}_\ell(Z) = \frac{1}{Z}\frac{\partial}{\partial Z}[Zj_\ell(Z)]$, and

$$\mathbf{M}_{\ell m}(\theta,\phi) = \hat{\boldsymbol{\theta}}\, \frac{im\, Y_{\ell m}(\theta,\phi)}{\sqrt{\ell(\ell+1)}\,\sin\theta} - \hat{\boldsymbol{\phi}}\, \frac{\frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\phi)}{\sqrt{\ell(\ell+1)}}, \quad \mathbf{N}_{\ell m}(\theta,\phi) = \hat{\mathbf{r}} \times \mathbf{M}_{\ell m}(\theta,\phi) \tag{9}$$

are the transverse vector-wave functions. The quantity $\mathbf{E}_p(ia_p\hat{\mathbf{r}})$ is the output of a complex-point receiver as explained in [3]; see also Section 2.3 below.

2.3. The Scattering Matrix

The spherical-wave scattering matrix $\bar{\mathbf{\Lambda}}$ (also referred to as Waterman's T-matrix) for a particular scatterer determines the spherical expansion coefficients $A_{\ell m}$ and $B_{\ell m}$ for the scattered field in terms of the spherical expansion coefficients $C_{\ell m}$ and $D_{\ell m}$ for the primary field as [4, 5] (note that [5] defines the scattering matrix in terms of "incoming" and "outgoing" spherical waves rather than the "standing" and "outgoing" spherical waves used here; see [5, Page 46])

$$A_{\ell m} = \sum_{\ell'=0}^{L} \sum_{m'=-\ell'}^{\ell'} \left[\Lambda^{(AC)}_{\ell\ell' mm'} C_{\ell' m'} + \Lambda^{(AD)}_{\ell\ell' mm'} D_{\ell' m'}\right], \quad B_{\ell m} = \sum_{\ell'=0}^{L} \sum_{m'=-\ell'}^{\ell'} \left[\Lambda^{(BC)}_{\ell\ell' mm'} C_{\ell' m'} + \Lambda^{(BD)}_{\ell\ell' mm'} D_{\ell' m'}\right]. \tag{10}$$

Hence, the scattered field can be expressed in terms of the output of elementary complex-point receivers as

$$\begin{aligned}\mathbf{E}_s(\mathbf{r}) = \int d\Omega'\, G\left(\mathbf{r}, ia_s\hat{\mathbf{r}}'\right) \Bigg[\mathbf{F}_M\left(\mathbf{r}, ia_s\hat{\mathbf{r}}'\right) \int d\Omega''\, \mathbf{Q}_A\left(\hat{\mathbf{r}}', \hat{\mathbf{r}}''\right) \cdot \mathbf{E}_p\left(ia_p\hat{\mathbf{r}}''\right) \\ + \mathbf{F}_N(\mathbf{r}, ia_s\hat{\mathbf{r}}') \int d\Omega''\, \mathbf{Q}_B(\hat{\mathbf{r}}', \hat{\mathbf{r}}'') \cdot \mathbf{E}_p(ia_p\hat{\mathbf{r}}'')\Bigg]\end{aligned} \tag{11}$$

where the complex-source scattering matrices are given by

$$\mathbf{Q}_A\left(\hat{\mathbf{r}}', \hat{\mathbf{r}}''\right) = \sum_{\ell=1}^{L} \sum_{m=-\ell}^{\ell} \frac{(ik)^{-1} Y_{\ell m}\left(\theta', \phi'\right)}{\sqrt{\ell\left(\ell+1\right)}\, j_\ell(ika_s)} \sum_{\ell'=0}^{L} \sum_{m'=-\ell'}^{\ell'} \left[\Lambda^{(AC)}_{\ell\ell' mm'} \frac{\mathbf{M}^*_{\ell' m'}\left(\theta'', \phi''\right)}{j_{\ell'}\left(ika_p\right)} + \Lambda^{(AD)}_{\ell\ell' mm'} \frac{\mathbf{N}^*_{\ell' m'}\left(\theta'', \phi''\right)}{v^{(1)}_{\ell'}\left(ika_p\right)}\right] \tag{12}$$

and $\mathbf{Q}_B(\hat{\mathbf{r}}', \hat{\mathbf{r}}'')$ is given by (12) with $\Lambda^{(AC)}_{\ell\ell' mm'}$ replaced by $\Lambda^{(BC)}_{\ell\ell' mm'}$ and $\Lambda^{(AD)}_{\ell\ell' mm'}$ replaced by $\Lambda^{(BD)}_{\ell\ell' mm'}$. One can derive integral equations from which the complex-source scattering matrices can be computed numerically for a given scatterer without using Waterman's T-matrix.

The Formula (11) expresses the scattered field directly in terms of the output $\mathbf{E}_p(ia_p\hat{\mathbf{r}}'')$ of an elementary complex-point receiver pointing in all directions (in other words, in terms of the incident electric field evaluated at imaginary points of the form $ia_p\hat{\mathbf{r}}''$, where $\hat{\mathbf{r}}''$ covers the unit sphere). The directivity of these receivers ensure that one can neglect directions $\hat{\mathbf{r}}''$ that do not point towards the primary source region. Similarly, the directivity of the transmitting beams in the $\hat{\mathbf{r}}'$ integral ensure that one can neglect directions $\hat{\mathbf{r}}'$ that do not point towards the observation region. Also, note that the scattering matrix contains two free parameters (the disk radii a_p and a_s) that can be used to optimize the efficiency of the scattering calculation. Numerical examples that demonstrate the directional nature of the scalar analog of this scattering-matrix formulation can be found in [6].

Let us briefly discuss how $\mathbf{E}_p(ia_p\hat{\mathbf{r}}'')$ can be obtained through simulations or measurements. If the software used to perform simulations defines the square root to have non-negative real part and its branch cut along the negative real axis (Matlab's square root is defined this way), one simply sets the observation point equal to the imaginary point $ia_p\hat{\mathbf{r}}''$. No further work is needed. If the software defines the square root differently, one must write a new square-root function defined as described above. If the primary field is measured on a sphere centered on the primary source, outgoing spherical expansion coefficients are known. One can then compute $\mathbf{E}_p(ia_p\hat{\mathbf{r}}'')$ directly from the spherical vector-wave function expansion.

3. EXPANSION BASED ON PLANE WAVES

We next derive a scattering-matrix formulation that is based on the directional plane-wave expansion from [7].

3.1. The Scattered Field

We begin by introducing the far-field pattern of the scattered field through the equation

$$\boldsymbol{\mathcal{F}}_s(\hat{\mathbf{r}}) = \lim_{r\to\infty} r\, e^{-ikr}\mathbf{E}_s(r\hat{\mathbf{r}}) \tag{13}$$

and find from [7, Eq. (52)] that

$$\mathbf{E}_s(\mathbf{r}) = \frac{ik}{4\pi}\int d\Omega_k\, \boldsymbol{\mathcal{F}}_s(\hat{\mathbf{k}})\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_o)}\, T_{N_o}\left(\hat{\mathbf{k}}, \mathbf{r}_o, \Delta_s\right) \tag{14}$$

where $\int d\Omega_k$ is the integral over the $\mathbf{k}$ unit sphere and

$$T_{N_o}\left(\hat{\mathbf{k}}, \mathbf{r}_o, \Delta_s\right) = e^{k\Delta\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}_o-1\right)}\sum_{n=0}^{N_o} i^n(2n+1)\tilde{h}_n^{(1)}(k\{r_o+i\Delta_s\})\, P_n\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}_o\right) \tag{15}$$

is the Gaussian translation operator that results in directional plane-wave spectra; see [7] for details. Further,

$$\tilde{h}_n^{(1)}(Z) = h_n^{(1)}(Z)\, e^{\mathtt{Im}(Z)} \tag{16}$$

is a normalized spherical Hankel function and $P_n(Z)$ is the Legendre polynomial. The truncation number N_o, which depends on the beam parameter Δ_s as well as on the size of the scatterer and observation region, can be found from the procedure in [7, Sec. VI] to achieve any desired accuracy. Note that the Gaussian translation operator (15) equals the standard translation operator when the beam parameter Δ_s equals zero.

3.2. The Primary Field

With the far-field pattern of the primary source (with respect to the origin $\mathbf{r}_p$) defined as

$$\boldsymbol{\mathcal{F}}_p(\hat{\mathbf{r}}) = \lim_{r\to\infty} r\, e^{-ikr}\mathbf{E}_p\left(r\hat{\mathbf{r}}-\mathbf{r}_p\right) \tag{17}$$

we find that

$$\mathbf{E}_p(\mathbf{r}) = \frac{ik}{4\pi}\int d\Omega_k\, \boldsymbol{\mathcal{F}}_p(\hat{\mathbf{k}})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, T_{N_p}\left(\hat{\mathbf{k}}, -\mathbf{r}_p, \Delta_p\right) \tag{18}$$

where the truncation number N_p, which depends on the beam parameter Δ_p as well as on the size of the scatterer and primary source region, can be found from the procedure in [7, Sec. VI]. According to (18), the plane-wave spectrum for the incoming primary field is $[(ik)/(4\pi)]\boldsymbol{\mathcal{F}}_p(\hat{\mathbf{k}})\, T_{N_p}(\hat{\mathbf{k}}, -\mathbf{r}_p, \Delta_p)$.

3.3. The Scattering Matrix

The far-field plane-wave scattering matrix $\bar{\mathbf{F}}(\hat{\mathbf{k}}', \hat{\mathbf{k}}'')$ determines the scattered far-field pattern in the direction $\hat{\mathbf{k}}'$ when the incident field is a single plane wave $\mathbf{E}_0\, e^{i\mathbf{k}''\cdot\mathbf{r}}$ (with the constant vector $\mathbf{E}_0$ satisfying $\mathbf{E}_0\cdot\mathbf{k}'' = 0$) as

$$\boldsymbol{\mathcal{F}}_s(\hat{\mathbf{k}}') = \bar{\mathbf{F}}(\hat{\mathbf{k}}', \hat{\mathbf{k}}'')\cdot\mathbf{E}_0 \tag{19}$$

which in turn gives the final expression for the scattered field

$$\mathbf{E}_s(\mathbf{r}) = -\frac{k^2}{16\pi^2}\int d\Omega_k'\, e^{i\mathbf{k}'\cdot(\mathbf{r}-\mathbf{r}_o)}\, T_{N_o}(\hat{\mathbf{k}}', \mathbf{r}_o, \Delta_s)\int d\Omega_k''\, T_{N_p}(\hat{\mathbf{k}}'', -\mathbf{r}_p, \Delta_p)\, \bar{\mathbf{F}}(\hat{\mathbf{k}}', \hat{\mathbf{k}}'')\cdot\boldsymbol{\mathcal{F}}_p(\hat{\mathbf{r}}''). \tag{20}$$

In general, a scatterer that is illuminated by a single plane wave will produce plane waves propagating in all directions. Hence, the scattering matrix Formula (20) contains an integral over both incoming plane-wave directions of propagation $\hat{\mathbf{k}}''$ and scattered plane-wave directions of propagation $\hat{\mathbf{k}}'$. The directionality of the Gaussian translation operators ensures that large portions of the $\hat{\mathbf{k}}'$ and $\hat{\mathbf{k}}''$ integration regions can be neglected.

The important special scattering geometry involving a half space was investigated in [8] using the standard translation operator ($\Delta_s = \Delta_p = 0$). The plane-wave scattering matrix for the half space is degenerate in the sense that a single incident plane wave will produce only one scattered plane wave. Therefore, for the half-space scatterer, the Formula (20) simplifies so that it involves only the integral over incoming (or outgoing) plane-wave directions of propagation.

4. CONCLUSIONS

The two scattering-matrix formulations in (11) and (20) will now be compared to Kerns' plane-wave scattering-matrix formulation (one-sided) in which the scattered field is written as [9, Pages 57–61]

$$\mathbf{E}_s(\mathbf{r}) = \int d\mathbf{K}'\, e^{i\mathbf{k}'\cdot\mathbf{r}} \int d\mathbf{K}'' \bar{\mathbf{S}}\left(\mathbf{K}', \mathbf{K}''\right) \cdot \mathbf{T}_p\left(\mathbf{K}''\right) \tag{21}$$

where $\bar{\mathbf{S}}(\mathbf{K}', \mathbf{K}'')$ is Kerns' plane-wave scattering matrix, and $\mathbf{T}_p(\mathbf{K}'')$ is the plane-wave spectrum of the primary field. Moreover, $\mathbf{K}'$ is the transverse part of the plane-wave propagation vector $\mathbf{k}'$ for the scattered field. Similarly, $\mathbf{K}''$ is the transverse part of the plane-wave propagation vector $\mathbf{k}''$ for the primary field. The plane-wave spectrum of the primary field $\mathbf{T}_p(\mathbf{K}'')$ can be thought of as the output of a plane-wave receiver that picks out a single plane-wave component of the primary field.

It is challenging to use the plane-wave expansion (21) in numerical computations that require a preselected accuracy. First, the integrand of the $\mathbf{K}'$ integral has an integrable singularity at $|\mathbf{K}'| = k$ that in many situations necessitates a change of variables [10, Chapter 3]. Second, in general the integrands do not decay until the evanescent regions $|\mathbf{K}'| > k$ and $|\mathbf{K}''| > k$ are reached. In some situations one must integrate all the way into the evanescent regions to avoid strong end-point contributions [7]. Nevertheless, there are some favorable situations where the integrals in (21) can be truncated to include only small regions in the propagating domains without introducing significant errors.

The expansions (11) and (20) have the same structure as (21). However, there are important differences. For example, the regions of integrations in both (11) and (20) are unit spheres whereas it is an infinite planar $\mathbf{K}$ surface in (21). Also, the integrands in (11) and (20) are well-behaved functions on the unit spheres that are directional at high frequencies, so that only parts of the unit spheres need to be included to achieve high accuracy. Hence, from a numerical point of view, the expansions (11) and (20) appear to have distinct advantages.

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