A Variational Method to Solve Maxwell's Equations in Singular Axisymmetric Domains with Arbitrary Data

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Abstract— We propose a variational method to solve Maxwell's equations in singular axisymmetric domains with arbitrary data. Considering the equations written in (r, θ, z), we use a Fourier transform in θ to reduce 3D equations to a series of 2D equations, depending on the Fourier variable k. We then consider the case $k = 0$, corresponding to the full axisymmetric case, and focus on the computation of the magnetic field. The non stationary variational formulation to compute the solution is derived, and solved with a finite element approach. Numerical examples are shown.

1. INTRODUCTION

We propose a new variational method to compute 3D Maxwell's equations in an axisymmetric singular 3D domain, generated by the rotation of a singular polygon around one of its sides, namely containing reentrant corner or edges. We consider the equations written in (r, θ, z) and use a Fourier transform in θ to reduce 3D Maxwell's equations to a series of 2D Maxwell's equations, depending on the Fourier variable k. The principle is to compute the 3D solution by solving several 2D problems, each one depending on k.

Let us denote by $(\mathbf{E}_k, \mathbf{B}_k)$ the electromagnetic field for each mode k. Following [2, 4], it can be proved that this solution can be decomposed into a regular and a singular part. The regular part can be computed with a classical finite element method. The singular part is more difficult to compute, as it belongs to a finite-dimensional subspace. Its dimension is equal to the number of reentrant corners and edges of the 2D polygon that generates the 3D domain by rotation. We will propose a variational approach, based on a decomposition of the computational domain into subdomains, and we will derive an ad hoc variational formulation.

We will derive the non stationary variational formulation to compute the solution. Following [4], this will require first to derive the system of equations solved by the singular parts for each k. Then to derive and solve the time-dependent variational formulation depending on k, for each mode. We will then detail the case $k = 0$, corresponding to the full axisymmetric case, and illustrate it with numerical results.

2. THE 3D MAXWELL EQUATIONS

Let Ω be a bounded and simply connected Lipschitz axisymmetric domain in $\mathbb{R}^3$, limited by the surface of revolution Γ. We denote by $\mathbf{n}$ the unit outward normal to Γ, and by ω and γ_b their intersections with a meridian half-plane (see Fig. 1). In the text, names of scalar quantities usually begin by an italic letter, whereas they begin by a bold letter for vector quantities. We denote by γ the boundary $\partial\omega = \gamma_a \cup \gamma_b$, where γ_a is the segment of the axis lying between the extremities of γ_b. We denote $\boldsymbol{\nu}$ is outward normal, and by $\boldsymbol{\tau}$ the unit tangential vector such that $(\boldsymbol{\tau}, \boldsymbol{\nu})$ is direct.

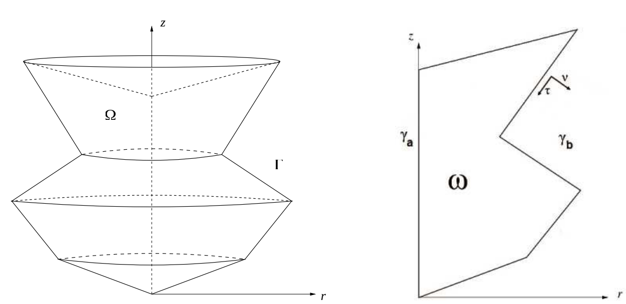

Figure 1: The Ω and ω domains.

The natural coordinates for this domain are the cylindrical coordinates (r, θ, z) with the basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$. A meridian half-plane is defined by the equations θ = constant. However, when symmetry of revolution is not assumed for the data, but only for the domain Ω, one can not reduce the problem to a two-dimensional one by assuming that $\partial/\partial\theta = 0$. For this reason, we continue to deal with a three-dimensional problem. The time-dependent Maxwell equations in vacuum can be expressed

$$\frac{\partial \mathbf{E}}{\partial t} - c^2 \mathbf{curl}\, \mathbf{B} = -\frac{1}{\varepsilon_0}\mathbf{J}, \qquad \mathrm{div}\, \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\, \mathbf{E} = 0, \qquad \mathrm{div}\, \mathbf{B} = 0\,, \tag{2}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic induction, and ρ and $\mathbf{J}$ are the charge and current densities. These quantities depend on the space variable $\mathbf{x}$ and on the time variable t. These equations are supplemented with perfect conductor boundary conditions, namely

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{and} \quad \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on the boundary } \Gamma.$$

together with homogeneous conditions at initial time $t = 0$, says $(\mathbf{E}, \mathbf{B})_{|t=0} = (\mathbf{E}_0, \mathbf{B}_0)$, for some given $(\mathbf{E}_0, \mathbf{B}_0)$. As explained in [1], it is worthwhile from a numerical point of view to write Maxwell's equations as second order formulations. Eliminating $\mathbf{E}$ and $\mathbf{B}$ between the evolutions equations of (1) and (2), one can replace them by

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} + c^2 \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = -\frac{1}{\varepsilon_0}\frac{\partial \mathbf{J}}{\partial t}, \tag{3}$$

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} + c^2 \mathbf{curl}\,\mathbf{curl}\,\mathbf{B} = \frac{1}{\varepsilon_0}\mathbf{curl}\,\mathbf{J}\,. \tag{4}$$

The divergence equations and boundary conditions still holding, and initial conditions for the time derivatives $\partial_t \mathbf{E}_{|t=0}, \partial_t \mathbf{B}_{|t=0}$ are supplemented, see details in [1]. In the following, we will concentrate on the magnetic field $\mathbf{B}$.

We now introduce the variational formulation of the problem. The function spaces are defined with classical notation. The usual scalar product of $\mathbf{L}^2(\Omega)$ are denoted by $(\cdot, \cdot)$. We shall also need to use the standard spaces and norms

$$\mathbf{H}(\mathbf{curl}\,, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega),\, \mathbf{curl}\,\mathbf{v} \in \mathbf{L}^2(\Omega)\},$$
$$\mathbf{H}(\mathrm{div}\,, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega),\, \mathrm{div}\,\mathbf{v} \in L^2(\Omega).$$

We also introduce $\mathbf{H}_0(\mathrm{div}\,; \Omega)$, the subspace of $\mathbf{H}(\mathrm{div}\,, \Omega)$, with a vanishing normal trace, that is $\mathbf{v} \cdot \mathbf{n}|_\Gamma = 0$. The magnetic field naturally belongs to the space

$$\mathbf{Y}\,(\Omega) = \mathbf{H}\,(\mathbf{curl}\,; \Omega) \cap \mathbf{H}_0\,(\mathrm{div}\,\mathbf{v}; \Omega)\ .$$

The scalar product on $\mathbf{Y}$ is defined as

$$a\,(\mathbf{u}, \mathbf{v}) := (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v})\,.$$

so that the variational formulation we have derived is written
Find $\mathbf{B} \in \mathbf{Y}\,(\Omega)$ *such that:*

$$\frac{d^2}{dt^2}\,(\mathbf{B}(t), \mathbf{C}) + c^2\, a\,(\mathbf{B}(t), \mathbf{C}) = \frac{1}{\varepsilon_0}\,(\mathbf{curl}\,\mathbf{J}, \mathbf{C})\,, \forall \mathbf{C} \in \mathbf{Y}\,(\Omega)\,. \tag{5}$$

3. TWO DIMENSIONAL REDUCTION

If we do not assume *a priori* that the data are axisymmetric, so we can not perform $\frac{\partial}{\partial\theta} = 0$. Hence, the problem, at this stage, can not be reduced to a two dimensional one. However, based on the symmetry of revolution of the domain Ω, we can characterize the electromagnetic fields by their Fourier series in θ. For a function $\mathbf{v}(r, \theta, z) \in \mathbf{L}^2(\Omega)$, one can define

$$\mathbf{v}\,(r, \theta, z) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \mathbf{v}^k\,(r, z)\, e^{ik\theta}$$

where $\mathbf{v}^k(r,z) = v_r^k\mathbf{e}_r + v_\theta^k\mathbf{e}_\theta + v_z^k\mathbf{e}_z$, so that for all $\mathbf{u},\mathbf{v} \in \mathbf{L}^2(\Omega)$, one has $(\mathbf{u},\mathbf{v}) = \sum_{k\in\mathbb{Z}}(\mathbf{u}^k,\mathbf{v}^k)_{\mathbf{L}^2_r(\omega)}$. The regularity of the Fourier coefficients $\mathbf{v}^k$ in Sobolev spaces defined on ω can be related to that of $\mathbf{v}$, in the corresponding spaces defined on Ω. Details can be found in [3, 4].

From this definition, one readily gets that

$$\mathbf{curl}\,\mathbf{v} = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\mathbf{curl}_k\mathbf{v}^k e^{ik\theta} \quad \text{and} \quad \operatorname{div}\mathbf{v} = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\operatorname{div}_k\mathbf{v}^k e^{ik\theta}, \tag{6}$$

where $\mathbf{curl}_k$ and div_k are defined by replacing the derivation with respect to θ by ik in the standard definition of $\mathbf{curl}$ and div. As a consequence, a function $\mathbf{v}$ belongs to $\mathbf{Y}(\Omega)$ if and only if, for all $k\in\mathbb{Z}$, its Fourier coefficients $\mathbf{v}^k$ belong to the space $\mathbf{Y}_{(k)}(\omega)$ defined by

$$\mathbf{Y}_{(k)}(\omega) = \{\mathbf{v}^k \in \mathbf{L}^2_r(\omega),\ \mathbf{curl}_k\mathbf{v}^k \in \mathbf{L}^2_r(\omega),\ \operatorname{div}_k\mathbf{v}^k \in \mathbf{L}^2_r(\omega),\ \mathbf{v}^k\cdot\mathbf{n}_{|\gamma_b} = 0\}.$$

Based on the linearity of the Maxwell equations and on the orthogonality of the Fourier modes for each k, one can readily derive from (6) that the magnetic field $\mathbf{B}$ is solution to (5), if its Fourier coefficients $\mathbf{B}_k$ verify the formulation, for each mode k

Find $\mathbf{B}^k \in \mathbf{Y}_{(k)}(\omega)$ *such that, for all* $\boldsymbol{C} \in \mathbf{Y}_{(k)}(\omega)$*:*

$$\frac{d^2}{dt^2}\Big(\mathbf{B}^k(t),\mathbf{C}\Big) + c^2\,a_k\Big(\mathbf{B}^k(t),\mathbf{C}\Big) = \frac{1}{\varepsilon_0}\Big(\mathbf{curl}_k\mathbf{J}^k,\mathbf{C}\Big)\,, \tag{7}$$

where

$$a_k(\mathbf{u},\mathbf{v}) = (\mathbf{curl}_k\mathbf{u},\mathbf{curl}_k\mathbf{v}) + (\operatorname{div}_k\mathbf{u},\operatorname{div}_k\mathbf{v})\,.$$

4. THE NUMERICAL METHOD

From now on, we will only consider the case $k=0$, corresponding to the full axisymmetric case. Also in this case, the reduced two dimensional domain ω remains singular (see Figure 1), and we have to deal with singularities. The construction of the numerical method is based on theoretical results proved in [7]. Indeed, the space of solution $\mathbf{Y}_{(0)}$ can be decomposed in

$$\mathbf{Y}_{(0)} = \mathbf{Y}^R_{(0)} \oplus \mathbf{Y}^S_{(0)}.$$

Above, $\mathbf{Y}^R_{(0)}$ is a regular subspace of $\mathbf{H}^1(\omega)$, which is the space of solutions in case of a regular domain, whereas $\mathbf{Y}^S_{(0)}$ is a singular subspace. As a consequence, the magnetic field solution $\mathbf{B}^0$ can be decomposed into a regular and a singular part, says

$$\mathbf{B}^0 = \mathbf{B}^0_R \oplus \mathbf{B}^0_S\,. \tag{8}$$

Moreover, the singular subspace is of finite dimension, the dimension of which depending on the number of singularities in the domain ω. For simplicity, we assume in what follows that there is one singularity. Let us introduce $\mathbf{y}^0_S$ the basis of $\mathbf{Y}^S_{(0)}$. Using that the basis is time independent, one gets that the singular part $\mathbf{B}^0_S$ can be decomposed into

$$\mathbf{B}^0_S = \mu(t)\mathbf{y}^0_S$$

where $\mu(t)$ is a smooth function in time. As a consequence, decomposition (8) can be expressed

$$\mathbf{B}^0(t) = \mathbf{B}^0_R(t) \oplus \mu(t)\mathbf{y}^0_S\,. \tag{9}$$

The computation method for the singular basis $\mathbf{y}^0_S$ was presented in [5] (see also [6] for the electric case). We present here the computation of the regular part $\mathbf{B}^0_R(t)$, and of the solution $\mathbf{B}^0(t)$.

Hence, we need to compute the values of $\mathbf{B}^0_R(t)$ and $\mu(t)$. Writing formulation (7) for $k=0$, using the *ad hoc* Green formula together with the boundary condition, the final variational formulation reads

Find $\mathbf{B}^0(t) \in \mathbf{Y}_{(0)}$ *such that:*

$$\frac{d^2}{dt^2}\left(\mathbf{B}^0(t),\mathbf{C}\right) + c^2 a_0\left(\mathbf{B}^0(t),\mathbf{C}\right) = \frac{1}{\varepsilon_0}\left(\mathbf{curl}_0\,\mathbf{J}^0,\mathbf{C}\right)\,,\forall\mathbf{C}\in\mathbf{Y}_{(0)}\,. \tag{10}$$

By taking into account that $\mathbf{B}^0(t) = \mathbf{B}^0_R(t) + \mu(t)\mathbf{y}^0_S$, this variational formulation can be expressed *Find* $\mathbf{B}^0_R \in \mathbf{Y}^R_{(0)}$ *and* $\mu(t)$ *such that:*

$$\frac{d^2}{dt^2}\left(\mathbf{B}^0_R(t),\mathbf{C}\right)+\mu''(t)\left(\mathbf{y}^0_S,\mathbf{C}\right)+c^2a_0\left(\mathbf{B}^0_R(t),\mathbf{C}\right)+c^2\mu(t)a_0\left(\mathbf{y}^0_S,\mathbf{C}\right)=\frac{1}{\varepsilon_0}\left(\mathbf{curl}\,_0\mathbf{J}^0,\mathbf{C}\right), \quad \forall\mathbf{C}\in\mathbf{Y}_{(0)}. \tag{11}$$

Writing the above variational formulation, first for regular test functions $\mathbf{C}_R \in \mathbf{Y}^R_{(0)}$, then for $\mathbf{y}^0_S \in \mathbf{Y}^S_{(0)}$, we finally obtain the system of equations

$$\begin{aligned}&\frac{d^2}{dt^2}\left(\mathbf{B}^0_R(t),\mathbf{C}_R\right)+\mu''(t)\left(\mathbf{y}^0_S,\mathbf{C}_R\right)+c^2\,a_0\left(\mathbf{B}^0_R(t),\mathbf{C}_R\right)+\ c^2\,\mu(t)\,a_0\left(\mathbf{y}^0_S,\mathbf{C}_R\right)\\ =\ &\frac{1}{\varepsilon_0}\left(\mathbf{curl}\,_0\,\mathbf{J}^0,\mathbf{C}_R\right),\forall\mathbf{C}_R\in\mathbf{Y}^R_{(0)}.\end{aligned} \tag{12}$$

$$\begin{aligned}&\frac{d^2}{dt^2}\left(\mathbf{B}^0_R(t),\mathbf{y}^0_S\right)+\mu''(t)\left(\mathbf{y}^0_S,\mathbf{y}^0_S\right)+c^2\,a_0\left(\mathbf{B}^0_R(t),\mathbf{y}^0_S\right)+c^2\,\mu(t)\,a_0\left(\mathbf{y}^0_S,\mathbf{y}^0_S\right)\\ =\ &\frac{1}{\varepsilon_0}\left(\mathbf{curl}\,_0\,\mathbf{J}^0,\mathbf{y}^0_S\right)\ ,\forall\mathbf{y}^0_S\in\mathbf{Y}^S_{(0)}\,.\end{aligned} \tag{13}$$

A conforming finite element method has been developed, based on the FreeFem++ package [8], to solve this problem. Numerical illustrations are given in the following section.

5. NUMERICAL RESULTS

We consider the 3-D top hat domain Ω with a reentrant circular edge, that corresponds to an L-shaped 2-D domain ω with a reentrant corner. We are interested in computing the magnetic field $\mathbf{B}^0(t)$ created by a current loop. In a first instance, we consider this domain and compute the basis of $\mathbf{y}^0_S$, on which a perfectly conducting boundary condition is imposed. Then, initial conditions are set to zero, and a current is defined as $\mathbf{J}^0(t) = 10\sin(\lambda t)\mathbf{e}_\theta$, with a frequency $\lambda/2\pi = 2.5$GHz. The support of this current is a little disc centered around the middle of the domain. This current generates a wave which propagates circularly around the current source. Physically, as long as the wave has not reached the reentrant corner, the field is smooth.

Let t_I be the impact time, then, if one writes $\mathbf{B}^0(t) = \mathbf{B}^0_R(t) + \mu(t)\mathbf{y}^0_S$, $\mu(t) = 0$ for all t lower than t_I, and $\mathbf{B}^0(t)$ and $\mathbf{B}^0_R(t)$ coincide. On the other hand, for $t > t_I$, $\mu(t) \neq 0$ (and so $\mu(t)\mathbf{y}^0_S$ is) and the total field differs from its regular part. This behavior is illustrated on Figures 2 and 3.

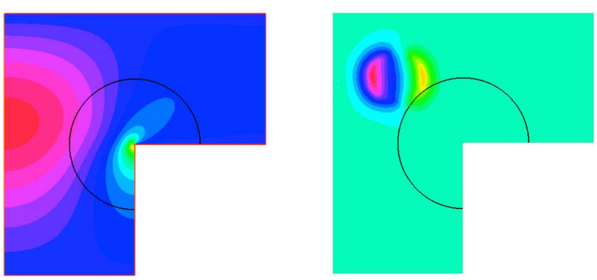

Figure 2: Singular basis $\mathbf{y}^0_S$ and $\mathbf{B}^0(t_1)$ for $t_1 < t_I$.

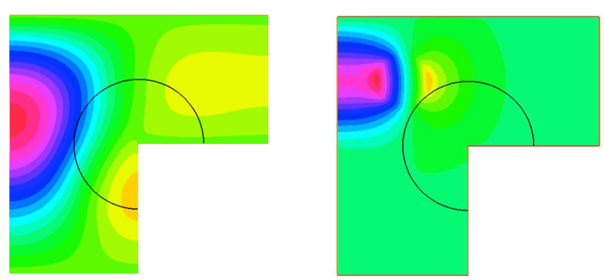

Figure 3: $\mathbf{B}^0(t_2)$ and $\mathbf{B}^0_R(t_2)$ for $t_2 > t_I$.

6. CONCLUSION

We have presented a variational formulation to solve the Maxwell equations in singular axisymmetric domains with arbitrary data. It is based on the use of a Fourier transform in θ to reduce 3D equations to a series of 2D equations, depending on the Fourier variable k. As an illustration, we completely derived the case $k = 0$, that corresponds to the full axisymmetric case. The approach was implemented and a numerical example was shown, to illustrate the feasibility of the method.

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