Whistler Modes Guided by Enhanced Density Ducts in a Nonresonant Magnetoplasma

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Abstract— Guidance of azimuthally symmetric waves by cylindrical density ducts in a magnetoplasma in the nonresonant region of the whistler frequency range is investigated. It is demonstrated that eigenmodes existing at the studied frequencies in ducts with enhanced density admit simplified description that makes analysis of the features of their guided propagation much easier. The results of calculation of the dispersion characteristics and field structures of the whistler modes supported by such ducts are reported.

1. INTRODUCTION

Investigation of the guided propagation of waves in the whistler frequency range in plasma channels (density ducts) in the presence of an external static magnetic field was performed in a large number of works (see [1, 2] and references therein). Whistler waves supported by such structures are intensely studied because of an important role that they play in many fundamental physical processes in the Earth's ionosphere and magnetosphere [1], as well as due to various applications including, in particular, use of the guiding properties of density ducts for controlling the characteristics of electromagnetic sources in a magnetoplasma [2], VLF diagnostics of the near-Earth space [3], etc. The overwhelming majority of the theoretical studies devoted to specific features of excitation and propagation of whistler waves in the presence of density ducts in a magnetoplasma deal with waves in the resonant region of the whistler range, which lies above the lower-hybrid resonance frequency $\omega_{\rm LH}$ and allows the existence of quasi-electrostatic waves [4–6]. It was shown in those works that in this frequency region, the ducts whose density is lower than the background value can support eigenmodes of different kinds, whereas the ducts with enhanced density can support no more than one eigenmode with the volume-surface field structure (for each fixed value of the azimuthal index) and improper leaky modes. As to the waves with frequencies below $\omega_{\rm LH}$, their guided propagation in cylindrical density ducts has attracted little attention in the literature.

It is the purpose of this work to study the dispersion properties and field structures of eigenmodes of density ducts in the case where the angular frequency ω of the field lies in the nonresonant region

$$\Omega_H < \omega < \omega_{\rm LH} \ll \omega_H \ll \omega_p \tag{1}$$

of the whistler range. Here, Ω_H is the ion gyrofrequency, and ω_H and ω_p are the gyrofrequency and the plasma frequency of electrons, respectively. Note that only an extraordinary wave having a closed refractive index surface is propagating in a magnetoplasma in the frequency range determined by inequalities (1). This circumstance leads to fundamental differences in the behavior of waves guided by the density ducts in this frequency interval compared with the case where the frequency ω lies in the resonant region of the whistler frequency range, in which the refractive index surface of the propagating extraordinary wave is open.

For simplicity, we limit our analysis to axisymmetric waves that are supported by cylindrical density ducts in range (1). Such guiding structures can appear in a magnetoplasma as a result of nonlinear interaction of the fields of electromagnetic sources with the surrounding (background) plasma medium. It should be noted that interest in specific features of the behavior of whistler waves in range (1) was stimulated in the past decade by a number of new experimental studies aimed at developing efficient methods of wave excitation in the indicated frequency interval [7].

2. MODES GUIDED BY A UNIFORM DENSITY DUCT

We consider an axisymmetric enhanced-density duct aligned with a uniform external magnetic field and surrounded by the cold collisionless background plasma. The external magnetic field $\mathbf{B}_0$ is assumed to be directed along the z axis of a cylindrical coordinate system (ρ, ϕ, z): $\mathbf{B}_0 = B_0\mathbf{z}_0$. In the presence of such a duct, the electron number (plasma) density N is a function of only the radial cylindrical coordinate ρ: $N = N(\rho)$. The plasma is described by the dielectric permittivity tensor with nonzero elements $\varepsilon_{\rho\rho} = \varepsilon_{\phi\phi} = \varepsilon$, $\varepsilon_{\rho\phi} = -\varepsilon_{\phi\rho} = -ig$, and $\varepsilon_{zz} = \eta$. Expressions for

ε, g, and η can be found elsewhere [2]. Fields of axisymmetric modes supported by the duct (with the time dependence $\exp(i\omega t)$ dropped) can be represented as $\mathbf{E}(\mathbf{r}) = \mathbf{E}(\rho)\exp(-ik_0pz)$ and $\mathbf{B}(\mathbf{r}) = \mathbf{B}(\rho)\exp(-ik_0pz)$, where k_0 is the wave number in free space and p is the normalized (to k_0) propagation constant. In this case, the quantities $\mathbf{E}(\rho)$ and $\mathbf{B}(\rho)$ can conveniently be described by scalar functions $E_\phi(\rho)$ and $E_z(\rho)$ that obey the equations

$$\frac{d^2E_\phi}{d\rho^2} + \frac{1}{\rho}\frac{dE_\phi}{d\rho} - \frac{E_\phi}{\rho^2} + k_0^2\left(\frac{g^2}{p^2-\varepsilon} - p^2 + \varepsilon\right)E_\phi = \frac{k_0pg}{p^2-\varepsilon}\frac{dE_z}{d\rho}, \tag{2}$$

$$\frac{d^2E_z}{d\rho^2} + \frac{1}{\rho}\frac{dE_z}{d\rho} + \frac{p^2}{(p^2-\varepsilon)\,\varepsilon}\frac{d\varepsilon}{d\rho}\frac{dE_z}{d\rho} + k_0^2\frac{\eta}{\varepsilon}(\varepsilon - p^2)\,E_z = k_0\frac{p}{\varepsilon}(p^2-\varepsilon)\frac{1}{\rho}\frac{d}{d\rho}\left(\frac{\rho g E_\phi}{p^2-\varepsilon}\right). \tag{3}$$

Note that in the presence of a uniform duct, the distribution of plasma density over the radius is given by the expression $N(\rho) = N_a + (\tilde{N} - N_a)[1 - U(\rho - a)]$, where U is a Heaviside step function, a is the duct radius, $N_a = \text{const}$ is the background plasma density ($\rho > a$), and $\tilde{N} = \text{const}$ is the plasma density inside the duct ($\rho < a$). In what follows, the quantities referring to the region inside the duct will be denoted by the tilde, whereas the quantities referring to the outer region will be written without such a designation.

In the case of a uniform duct, the field components of the axisymmetric modes can be described by general expressions obtained in [2]. Outside the duct ($\rho > a$), expressions for the azimuthal and longitudinal electric-field components, which are solutions of Eqs. (2) and (3), can be written as follows:

$$E_\phi = i\sum_{k=1}^{2} C_k\,K_1(k_0s_k\rho), \quad E_z = -\frac{i}{\eta_a}\sum_{k=1}^{2} C_kn_ks_k\,K_0(k_0s_k\rho), \tag{4}$$

where K_m is the mth-order modified Bessel function of the second kind, and C_1 and C_2 are constants. The quantities s_k and n_k are determined by the relations

$$\begin{gathered} s_k^2 = -q_k^2, \quad n_k = -\varepsilon[p^2 + q_k^2 + (g^2-\varepsilon^2)\varepsilon^{-1}](pg)^{-1}, \\ q_k = \left\{[\varepsilon^2 - g^2 + \varepsilon\eta - (\varepsilon+\eta)p^2 + (-1)^kR(p)](2\varepsilon)^{-1}\right\}^{1/2}, \\ R(p) = (\varepsilon-\eta)[(p^2-P_b^2)(p^2-P_c^2)]^{1/2}, \\ P_{b,c} = \left\{\varepsilon - (\varepsilon+\eta)\frac{g^2}{(\varepsilon-\eta)^2} + \frac{2\chi_{b,c}}{(\varepsilon-\eta)^2}[\varepsilon g^2\eta\,(g^2-(\varepsilon-\eta)^2)]^{1/2}\right\}^{1/2}, \quad \chi_b = -\chi_c = -1. \end{gathered} \tag{5}$$

Inside the duct ($\rho < a$), the azimuthal and longitudinal electric-field components are represented as

$$E_\phi = i\sum_{k=1}^{2} B_k\,J_1(k_0q_k\rho), \quad E_z = \frac{i}{\tilde{\eta}}\sum_{k=1}^{2} B_k\,\tilde{n}_k\,q_k\,J_0(k_0q_k\rho), \tag{6}$$

where J_m is the Bessel function of order m, B_1 and B_2 are constants, and the quantities q_k and $\tilde{n}_k$ can be obtained from the corresponding expressions for q_k and n_k in (5) by putting $\varepsilon = \tilde{\varepsilon}$, $g = \tilde{g}$, and $\eta = \tilde{\eta}$ (when doing this, the quantities P_b and P_c are replaced by the corresponding quantities $\tilde{P}_b$ and $\tilde{P}_c$, respectively).

The continuity condition for the tangential field components at $\rho = a$ yields a rigorous dispersion relation that allows determining the propagation constants p of the considered modes. Such a relation is rather cumbersome and can be analyzed only numerically. However, it admits an approximate analytical solution at $\Omega_H \ll \omega < \omega_{\mathrm{LH}}$. In this case, after some lengthy algebra we obtain the approximate dispersion relation

$$\frac{J_1(k_0q_1a)}{J_0(k_0q_1a)} = -\left(1 - \frac{\tilde{P}^2}{ps_1\sqrt{u_{\mathrm{LH}}-1}}\frac{K_1(k_0s_1a)}{K_0(k_0s_1a)}\right)^{-1}\frac{q_1K_1(k_0s_1a)}{s_1K_0(k_0s_1a)}, \tag{7}$$

where $u_{\mathrm{LH}} = \omega_{\mathrm{LH}}^2/\omega^2$ and $\tilde{P} = (\tilde{\varepsilon} - \tilde{g})^{1/2}$.

The dependences $p(\omega)$ satisfying the rigorous dispersion relation and approximate relation (7) are illustrated in Fig. 1(a). In this case, the duct supports the guided propagation only of the lowest

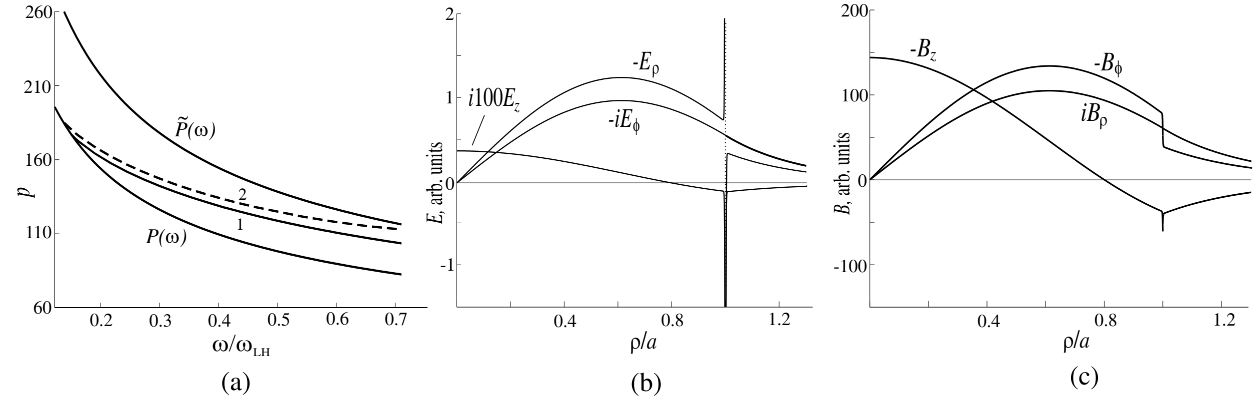

Figure 1: (a) Dispersion curves $P(\omega)$ of the lowest axisymmetric mode, which were calculated using the rigorous dispersion relation (curve 1) and approximate relation (7) (curve 2), and the dependences $\tilde{P}(\omega)$ and $P(\omega)$ for $\tilde{\omega}_p/\omega_{pa} = 1.41$, $\omega_{pa}/\omega_H = 4.5$, $\omega_H/\omega_{\mathrm{LH}} = 235$, and $\omega_{\mathrm{LH}}a/c = 5.25 \times 10^{-2}$. Transverse distributions of the electric (b) and magnetic (c) field components of this mode at $\omega/\omega_{\mathrm{LH}} = 0.68$.

eigenmode with the radial index $n = 1$. It is seen in the figure that the frequency dependences of the propagation constant obtained by rigorous and approximate methods are almost identical. The only exception is the vicinity of the lower hybrid resonance frequency ω_{LH}. When the frequency ω increases and approaches the lower hybrid resonance frequency, the accuracy of the approximate solution considerably deteriorates. With decreasing frequency, the mode propagation constant approaches the lower boundary $p = P(\omega) = (\varepsilon - g)^{1/2}$. After crossing this boundary, the mode ceases to be localized and becomes leaky.

The distributions of the field components of the lowest axisymmetric mode along the radial coordinate are illustrated in Figs. 1(b) and 1(c). The propagation constant of this mode $p = 104.87$. In this case, the quantities s_1 and s_2, which determine the field outside the duct, are real and the mode turns out to be localized. Inside the duct, q_1 is real, while q_2 is purely imaginary. The presence of the branches $q_2 = i\tilde{s}_2$ and s_2 leads to the appearance of sharp spikes in the components E_ρ and E_z and relatively fast changes in the components B_ϕ and B_z near the duct boundary. We emphasize that contribution of the components related to the presence of the branches $\tilde{s}_2$ and s_2 becomes evident only in close proximity to the duct boundary because of the large difference between the values of the transverse wave numbers q_1 and s_1 on the one hand, and $\tilde{s}_2$ and s_2 on the other hand. It can be seen in Figs. 1(b) and 1(c) that the lowest mode ($n = 1$) is characterized by the presence of one field oscillation over the radius, and the field structure is everywhere determined by the whistler component corresponding to the transverse wave numbers q_1 and s_1 and having quasi-TE polarization. In what follows, we will demonstrate that this circumstance allows one to substantially simplify the description of the fields of eigenmodes in the nonresonant region of the whistler frequency range, which appreciably facilitates their analysis in the case of a duct with a smooth variation in plasma density in the transverse direction.

3. MODES GUIDED BY A NONUNIFORM DENSITY DUCT

We now analyze the propagation of waves of the whistler range in a duct in which plasma density is described by a smooth function $N(\rho)$ monotonically decreasing from its maximum value $\tilde{N} = N(0)$ to the background value N_a with increasing cylindrical coordinate ρ. As in the preceding section, we will use the notations $\tilde{P}$ and P corresponding to the plasma densities $N = \tilde{N}$ and $N = N_a$, respectively.

Let us introduce the following notations:

$$\hat{q}_1^2 = \frac{g^2}{p^2 - \varepsilon} - p^2 + \varepsilon, \quad \hat{q}_2^2 = -\frac{\eta}{\varepsilon}\left(p^2 - \varepsilon\right), \quad \chi = \left(k_0 a p u^{1/2}\right)^{-1}, \quad \xi = \rho/a. \tag{8}$$

Hereafter, a denotes the characteristic transverse scale of the plasma density drop in a nonuniform duct, while $\hat{q}_1$ and $\hat{q}_2$, in contrast to the corresponding quantities in the preceding section, are functions of the coordinate ρ. In new notations, Eqs. (2) and (3) for axisymmetric modes take the form

$$\frac{d^2E_\phi}{d\xi^2}+\frac{1}{\xi}\,\frac{dE_\phi}{d\xi}-\frac{E_\phi}{\xi^2}+\left(k_0a\hat{q}_1\right)^2E_\phi=k_0ap\,\frac{g}{p^2-\varepsilon}\,\frac{dE_z}{d\xi},\tag{9}$$

$$\chi^2(1-u_{\rm LH})\left(\frac{d^2E_z}{d\xi^2}+\frac{1}{\xi}\,\frac{dE_z}{d\xi}+\frac{p^2}{(p^2-\varepsilon)\varepsilon}\,\frac{d\varepsilon}{d\xi}\,\frac{dE_z}{d\xi}\right)-\frac{\varepsilon}{p^2\eta}\,\hat{q}_2^2\,E_z=-\chi\,\frac{p^2-\varepsilon}{g}\,\frac{1}{\xi}\,\frac{d}{d\xi}\,\left(\frac{\xi gE_\phi}{p^2-\varepsilon}\right).\tag{10}$$

Allowing for the condition $P<p<\tilde{P}$ and the inequality $k_0ap\geq1$, which holds for the considered axisymmetric whistler modes in an enhanced-density duct, it can easily be verified that the following relations are satisfied in the case under consideration: $\chi\ll1$, $\chi u_{\rm LH}^{1/2}\ll1$, $p^2\gg|\varepsilon|$, and $|\varepsilon\hat{q}_2^2/p^2\eta|\simeq1$, which allows using the perturbation method for analysis of Eqs. (9) and (10). In zero-order perturbation theory ($\chi=0$ and $\chi^2u_{\rm LH}=0$), we have

$$\frac{d^2E_\phi}{d\rho^2}+\frac{1}{\rho}\,\frac{dE_\phi}{d\rho}-\frac{E_\phi}{\rho^2}+k_0^2\left(\frac{g^2}{p^2}-p^2\right)E_\phi=0\tag{11}$$

and $E_z=0$. The nontrivial expression for the longitudinal component of the electric field is obtained from Eq. (10) in the next (first-order) approximation of the perturbation theory ($\chi\neq0$ and $\chi^2\,(1-u_{\rm LH})\approx0$) and has the form

$$E_z=-\frac{1}{k_0p\eta}\,\frac{1}{\rho}\,\frac{d}{d\rho}\,(\rho gE_\phi).\tag{12}$$

In the zero-order approximation, other field components are expressed as

$$E_\rho=-igp^{-2}E_\phi,\quad B_\rho=-pE_\phi,\quad B_\phi=-igp^{-1}E_\phi,\quad B_z=\frac{i}{k_0\rho}\frac{d}{d\rho}(\rho E_\phi).\tag{13}$$

In the case of a weakly nonuniform cylindrical density duct, the density profile in the vicinity of the axis can be approximated by the quadratic law

$$N(\rho)=\tilde{N}\left(1-\frac{\rho^2}{2a^2}\right).\tag{14}$$

In this case, in Eq. (11) one should put $g^2\approx\tilde{g}^2\,(1-\rho^2/a^2)$, where $\tilde{g}=g(0)$. The solution $E_\phi(\rho)$ of Eq. (11) must be finite everywhere and approach zero as ρ tends to infinity ($\rho\to\infty$). Let us seek the solution for the component E_ϕ in the form

$$E_\phi=\zeta^{1/2}f(\zeta)\exp\left(-\frac{\zeta}{2}\right),$$

where $\zeta=\rho^2k_0\,|\tilde{g}|/(pa)$. Substituting this representation into Eq. (11) yields the equation

$$\zeta f''(\zeta)+(2-\zeta)\,f'(\zeta)+(n-1)\,f(\zeta)=0,$$

where $n=(\tilde{g}^2-p^4)\,k_0a/(4\,|\tilde{g}|\,p)$. Solution to this equation yields the function $E_\phi(\rho)$ satisfying all the above-mentioned conditions only if n is positive integer. As a result, we obtain the following dispersion relation for the propagation constant p:

$$p^4+\frac{4n|\tilde{g}|}{k_0a}\,p-\tilde{g}^2=0,\quad n=1,2,\ldots\,.\tag{15}$$

In this case, the solution $f(\zeta)$ is expressed in terms of a confluent hypergeometric function $\Phi(a,\,c;\,x)$: $f(\zeta)=\Phi(1-n,\,2;\,\zeta)$ [8]. Taking into account that the function $\Phi(-k,\,1+\alpha;\,x)$ for nonnegative integer k is reduced to the generalized Laguerre polynomials [8]

$$L_k^\alpha(x)=\frac{1}{k!}\,e^xx^{-\alpha}\,\frac{d^k}{dx^k}\,(e^{-x}x^{\alpha+k}),$$

the component E_ϕ of the field of the mode with the number n can be expressed in the form

$$E_\phi = E_0 \frac{\rho}{\rho_n} L_{n-1}^1 \left(\frac{\rho^2}{\rho_n^2}\right) \exp\left(-\frac{\rho^2}{2\rho_n^2}\right), \tag{16}$$

where E_0 is an arbitrary constant, $\rho_n = (p_n a / k_0 |\tilde{g}|)^{1/2}$ is the characteristic transverse scale of the field distribution of the nth mode over the radius, and p_n is the propagation constant which is the solution of Eq. (15). For the lowest mode, i.e., for $n = 1$, azimuthal component (16) is written as

$$E_\phi = E_0 \frac{\rho}{\rho_1} \exp\left(-\frac{\rho^2}{2\rho_1^2}\right). \tag{17}$$

All other components can be obtained from the azimuthal component of the electric field.

4. CONCLUSION

In this work, we have studied the dispersion properties and field structures of axisymmetric eigenmodes guided by density ducts in a magnetoplasma in the nonresonant region of the whistler frequency range. It is established that eigenmodes in this frequency region exist only in the ducts in which plasma density is higher than the background density. The possibility of a simplified description of these modes due to the extremely small value of the longitudinal component of their electric field inside and outside the duct is demonstrated. It is important that this possibility exists for both a sharp-walled uniform duct and a smooth-walled nonuniform duct. Obviously, such a simplified description of the modes should make analysis of the features of their excitation by electromagnetic sources much easier. This circumstance should be taken into account when solving the corresponding problems, in particular, when the source is located in the near-axis region of the duct.

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