A New Approach to Diffraction in Volume Gratings and Holograms

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Abstract— Coupled wave theory has historically been successfully used to provide various analytic theories of optical diffraction in volume phase gratings. Here we develop a different approach based on a decomposition of the permittivity profile of the grating into an infinite array of infinitesimal discontinuities or step functions. By considering the corresponding elementary Fresnel solutions from each discontinuity, a first-order set of coupled partial differential equations can be derived and then solved in a rotated frame of reference to give analytical expressions for the diffraction efficiency of the general slanted grating at arbitrary angle of incidence. The underlying differential equations derived from the approach are a rigorous representation of Maxwell's equations for the case of the unslanted grating and few assumptions are required to provide highly accurate solutions for harmonic or quasi-harmonic permittivity distributions, even at large grating slant.

1. INTRODUCTION

Historically coupled wave theory has been successfully used to provide analytic theories of optical diffraction in volume phase gratings [1–3]. Kogelnik's theory [1] has been particularly successful in this regard and indeed even today engineers frequently use his analytic expressions. Rigorous coupled wave theory [3] provides an exact estimate of the diffraction efficiency of a volume grating, but here numerical evaluation is required and in some cases such evaluations can be lengthy and subject to instability.

Here we shall describe a different approach — the PSM or Parallel Stacked Mirror model [4–11]. The PSM model is based on a decomposition of either the permittivity profile or the refractive index profile of the grating into an infinite array of infinitesimal discontinuities or step functions. By considering the corresponding elementary Fresnel solutions from each discontinuity, a first-order set of coupled partial differential equations can be derived. These equations are then solved in a rotated frame of reference to give analytical expressions for the diffraction efficiency of the general slanted grating at arbitrary angle of incidence. The underlying differential equations derived from this approach are a rigorous representation of Maxwell's equations for the case of the unslanted grating. Few assumptions are required to provide highly accurate solutions for the case of harmonic or quasi-harmonic permittivity distributions, even at large grating slant.

The method can be applied to either reflection or transmission geometry, although it is most accurate in the case of the reflection grating. It can also been extended to include finite absorption in phase gratings and to treat the cases of absorption and mixed absorption-phase gratings. It can also be used to treat multi-chromatic gratings, spatially multiplexed gratings and holograms. Finally it can be used to provide accurate analytic expressions for gratings with variable fringe contrast profiles.

Comparison with rigorous numerical computations of Maxwell's equations show that the approach works somewhat better than simple coupled wave theory for most reflection gratings, whereas the coupled wave approach appears a little better in describing the transmission geometry. The approach appears to offer a general approximate method for the analysis of electromagnetic, acoustic and other types of diffraction occurring in harmonic or quasi-harmonic structures.

2. USING FRESNEL REFLECTION TO ANALYSE DIFFRACTION

When a plane light wave propagates across the boundary of two regions of constant but differing refractive index, a portion of the light wave is reflected and a portion is transmitted. In the case of a purely real index there is no absorption and the transmitted and reflected energies add up to the incident energy. The laws describing how the amplitude of the transmitted and reflected waves depends on the indices of the two regions are known as Fresnel's laws. In the case of normal incidence they are trivially simple. If the wave passes from index n_1 to index n_2 then the ratio of the reflected to incident amplitude and the ratio of transmitted to incident amplitude are simply given by

$$r = \frac{n_2 - n_1}{n_1 + n_2}; \quad t = \frac{2n_1}{n_1 + n_2} \tag{1}$$

The PSM theory models a volume grating as many thin parallel stacked layers, each with a slightly different index. Overall the grating looks just like a normal grating except that under the microscope the index does not vary smoothly from one depth to another but rather it makes small jumps. However, by making the distance between such jumps vanishingly small, we won't know the difference.

When an illuminating wave strikes a jump between one index and the next, we can use Equation (1) to write down the amount of light transmitted and reflected. Of course these equations are only valid for the case of normal incidence but we can generalise them easily enough to incidence at any angle. Since the index layers are assumed flat and parallel in the simplest PSM model we can also use the well-known law that the angle of incidence is equal to the angle of reflection. If we then call our (monochromatic) illuminating wave, R, it is immediately obvious that at each index jump, R is slightly depleted and a new wave, which we can call S, is created by Fresnel reflection.

3. THE SIMPLEST POSSIBLE PSM THEORY — THE NORMAL-INCIDENCE VOLUME REFLECTION PHASE GRATING

We will now present the simplest possible mathematical derivation of the simplest example of PSM. This is the case of the sinusoidal unslanted lossless volume reflection phase grating under illumination at normal incidence. Fig. 1 shows a diagram of the set-up. We start by modelling the grating as a stack of parallel slices, each having a slightly differing refractive index. At this stage we don't make any assumption about the form of the index distribution other than it being composed of slices. Note that each slice will have a small but finite thickness and in the other two dimensions it will have the form of an infinite rectangular plane. We now assume that a monochromatic plane wave illuminates the grating. Mathematically we state this by the equation

$$R^{Ext} = e^{i\beta y} \tag{2}$$

where

$$\beta = \frac{2\pi n_0}{\lambda_c} \tag{3}$$

Here n_0 is the average value of index inside and outside of the grating and λ_c is the free-space wavelength. If we now label each one of the slices by the integer J and let the index of the Jth slice be n_J then we can immediately write down the following recurrence relations

$$\begin{aligned} R_J &= 2e^{i\beta\delta y}\left\{\frac{n_{J-1}}{n_J+n_{J-1}}\right\}R_{J-1} + e^{i\beta\delta y}\left\{\frac{n_{J-1}-n_J}{n_J+n_{J-1}}\right\}S_J \\ S_J &= 2e^{i\beta\delta y}\left\{\frac{n_{J+1}}{n_{J+1}+n_J}\right\}S_{J+1} + e^{i\beta\delta y}\left\{\frac{n_{J+1}-n_J}{n_{J+1}+n_J}\right\}R_J \end{aligned} \tag{4}$$

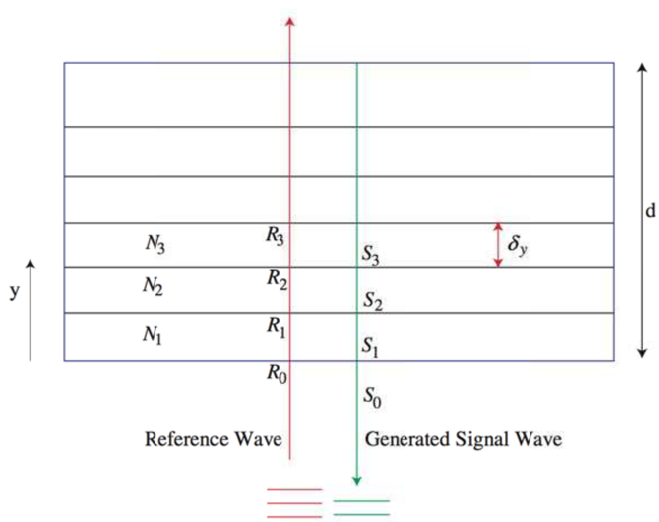

Figure 1: Normal Incidence grating with illuminating wave R and provoked signal wave, S.

Here the terms in curly brackets are just the Fresnel amplitude reflection and transmission coefficients and the exponential is a phase propagator which advances the phase of the R and S waves as the two waves travel the distance δy between discontinuities. We now let

$$X_{J-1} = X_J - \frac{dX}{dy}\delta y - \dots \tag{5}$$

and let $\delta y \to 0$. Here X denotes R, S or n. Further expanding the exponential terms as Taylor series and ignoring quadratic terms in δy we arrive at the differential counterpart to (4)

$$\begin{aligned}\frac{dR}{dy} &= \frac{R}{2}\left(2i\beta - \frac{1}{n}\frac{dn}{dy}\right) - \frac{1}{2n}\frac{dn}{dy}S\\ \frac{dS}{dy} &= -\frac{S}{2}\left(\frac{1}{n}\frac{dn}{dy} + 2i\beta\right) - \frac{1}{2n}\frac{dn}{dy}R\end{aligned} \tag{6}$$

These equations, which are not surprisingly an exact representation of Maxwell's equations, can be solved to produce approximate but very accurate analytic expressions for R and S and indeed for the diffraction efficiency of the grating in the case of a sinusoidal index:

$$n = n_0 + n_1\cos\left(\frac{4\pi n_0}{\lambda_r}y\right) = n_0 + \frac{n_1}{2}\left\{e^{\frac{4i\pi n_0}{\lambda_r}y} + e^{-\frac{4i\pi n_0}{\lambda_r}y}\right\} \tag{7}$$

Here λ_r is the free-space recording wavelength. We may write the diffraction efficiency for such a grating as

$$\eta_{PSM} = \frac{\alpha^2\kappa^2}{\beta^2(1-\alpha)^2 + \left(\alpha^2\kappa^2 - \beta^2(1-\alpha)^2\right)\coth^2\left(d\sqrt{\alpha^2\kappa^2 - \beta^2(1-\alpha)^2}\right)} \tag{8}$$

where

$$\alpha = \lambda_c/\lambda_r \quad \text{and} \quad \kappa = \frac{\pi n_1}{\lambda_c} \tag{9}$$

For comparison, Kogelnik's theory gives

$$\eta_{Kog} = \frac{\kappa^2}{\beta^2\alpha^2(1-\alpha)^2 + (\kappa^2 - \beta^2\alpha^2(1-\alpha)^2)\coth^2\left(d\sqrt{\kappa^2 - \beta^2\alpha^2(1-\alpha)^2}\right)} \tag{10}$$

Practically, the two expressions produce very similar results. However the PSM theory describes the behaviour away from Bragg resonance rather better.

4. EXTENSION TO OBLIQUE INCIDENCE AND FINITE LOSS

An unslanted lossy grating can be characterized by a harmonic refractive index

$$n = (n_0 + i\chi_0) + \frac{1}{2}(n_1 + i\chi_1)\left\{e^{i\mathbf{K}\cdot\mathbf{r}} + e^{-i\mathbf{K}\cdot\mathbf{r}}\right\} \tag{11}$$

where n_0 is the real average refractive index of the grating, n_1 is the real harmonic index modulation, χ_0 is the average imaginary index of the grating, χ_1 is the imaginary harmonic index modulation and $\mathbf{K}$ is the grating vector.

The grating is illuminated by a wave at oblique incidence

$$R = e^{ik_{cx}x + ik_{cy}y} \tag{12}$$

where

$$\mathbf{k}_c = \frac{2\pi n_0}{\lambda_c}\begin{pmatrix}\sin\theta_c\\ \cos\theta_c\end{pmatrix} = \beta\begin{pmatrix}\sin\theta_c\\ \cos\theta_c\end{pmatrix} \tag{13}$$

The grating vector, $\mathbf{K}$ can be written in terms of a recording angle θ_r and a recording free-space wavelength λ_r

$$\mathbf{K} = 2\alpha\beta\cos\theta_r\hat{\mathbf{y}} \tag{14}$$

The response of the grating to illumination is the generation of a reflected signal wave

$$S = S(y)e^{ik_{ix}x+ik_{iy}y} \tag{15}$$

with

$$\mathbf{k}_i = \beta\begin{pmatrix}\sin\theta_c\\ -\cos\theta_c\end{pmatrix} \tag{16}$$

The grating, as shown in Fig. 2, is divided into an infinite number of parallel stacked mirrors as before. At each such mirror the complex index makes a discontinuous jump and we may apply Fresnel's law to write down expressions for the amplitude transmission and reflection coefficients. Using the notation of superscripts to indicate mirror number in y and subscripts to indicate the quantised x position of a ray intersection, the transmission (t) and reflection (r) amplitude coefficients for the mirror between dielectric regions k and $k+1$ are respectively

$$\begin{pmatrix}t\\ r\end{pmatrix}_{k,k+1} = \frac{1}{\left\{\begin{matrix}(n_{k+1}+i\chi_{k+1})\sqrt{1-\frac{n_0^2}{(n_{k+1}+i\chi_{k+1})^2}\sin^2\theta_c}\\ +(n_k+i\chi_k)\sqrt{1-\frac{n_0^2}{(n_k+i\chi_k)^2}\sin^2\theta_c}\end{matrix}\right\}}\begin{pmatrix}2(n_k+i\chi_k)\sqrt{1-\frac{n_0^2}{(n_k+i\chi_k)^2}\sin^2\theta_c}\\ \left\{\begin{matrix}(n_{k+1}+i\chi_{k+1})\sqrt{1-\frac{n_0^2}{(n_{k+1}+i\chi_{k+1})^2}\sin^2\theta_c}\\ -(n_k+i\chi_k)\sqrt{1-\frac{n_0^2}{(n_k+i\chi_k)^2}\sin^2\theta_c}\end{matrix}\right\}\end{pmatrix} \tag{17}$$

As before coupled recurrence relations can now be derived for R and S and from these a set of coupled partial differential equations [9]. These can then be solved to define approximate but nevertheless very accurate analytic expressions for the diffraction efficiency. The coupled partial differential equations may also be solved in a rotated frame, thereby describing gratings with arbitrary slant and indeed even transmission gratings [4]. A recent study [10] has shown that the PSM model provides a generally better description of the typical reflection grating than traditional coupled wave theory, although the converse appears to be true for the transmission grating. PSM can also be extended to panchromatic gratings [4], multiplexed gratings and holograms [5] and variable contrast gratings [11]. The PSM model may additionally be utilised to **describe particle** diffraction from quantum periodic structures as the time independent Schrödinger equation for a harmonic potential is analytically identical to the corresponding Helmholtz equation describing optical diffraction from a harmonic index. Potential applications include the analysis of neutron super-mirrors that have been recorded using holographic techniques [12, 13]. Finally the results may also be useful in the study of acoustic diffraction from harmonic structures where the transfer matrix approach is well known [14].

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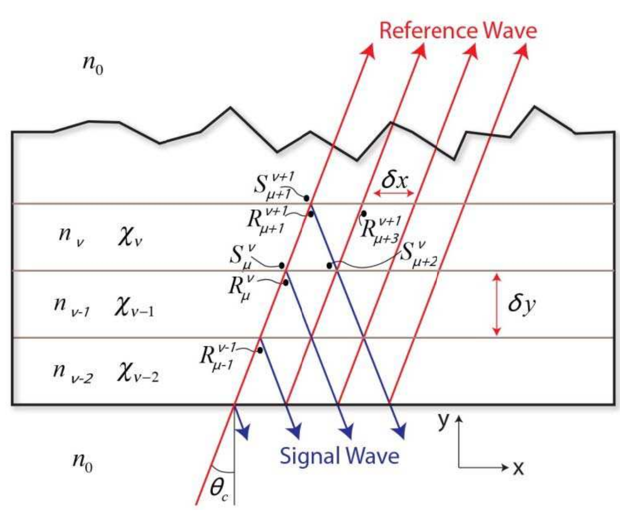

Figure 2: The Replay of an unslanted reflection grating, as treated by the PSM model, showing infinitesimally thick dielectric layers and the Signal and Reference fields present at the index discontinuities.

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