Unified Description of Chirped Gaussian Pulse Propagation of Arbitrary Initial Width in a Multiple Resonance Lorentz Medium

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Abstract— The unified asymptotic approach yields an accurate description of chirped Gaussian pulse propagation in a causally dispersive and absorptive multiple resonance Lorentz medium, and is valid from the slowly-varying envelope to the sub-cycle pulse regimes. The dynamical evolution of the propagated field is directly related to the dynamics of the saddle points of the unified complex phase function. Accordingly, the (total) propagated field is comprised of pulse components each being due to the contribution of a respective relevant saddle point. The instantaneous oscillation frequency and the occurrence/propagation velocity of a stationary point of the envelope, of each such pulse component are intrinsically related with the dynamics of the respective relevant saddle point. Under certain conditions, a stationary point of the envelope of a pulse component occurs when the respective saddle point trajectory intersects the real frequency axis, and it travels with the group velocity at this frequency crossing value.

1. INTRODUCTION AND FORMULATION OF THE PULSE PROPAGATION PROBLEM

Research interest on ultrashort pulse-modulated signals, especially those with duration comparable to the carrier oscillation cycle, is currently increasing and extends from the RF to the XUV [1–20]. Particular emphasis is placed on the dynamical evolution of such ultrashort pulses as they propagate through, and interact with, various material media in view of their applications, among others, in optical communications, radar systems and pulse compression [1, 2, 10–13, 15–18]. Related theoretical efforts commonly invoke the slowly varying envelope approximation (SVEA) [1, 2, 4, 6–9, 11–14, 16] and are inapplicable in the sub-cycle pulse regime [1, 2, 4, 6, 8, 9, 12, 13, 16, 19]. A different stream of rigorous theoretical efforts to these pulse propagation problems employs asymptotic techniques [2–5, 8, 13, 20]. The latter efforts have elucidated the dynamical evolution of the propagated field due to rapid rise and/or fall time or exponentially varying input pulses and their accuracy has been validated upon comparison with purely numerical results.

Herein, the unified asymptotic approach is applied in order to obtain an accurate description of the dynamical evolution of the propagated field in chirped Gaussian pulse propagation of arbitrary initial width in a causal, multiple resonance Lorentz-type dielectric. The starting point is the unified, exact integral expression

$$A(z,t) = \frac{1}{2\pi}\mathrm{Re}\left\{ i\int_L \tilde{U}_U \exp\left[\frac{z}{c}\Phi_U(\omega,\theta)\right] d\omega \right\} \tag{1}$$

which describes the propagated field $A(z,t)$ in the positive z-direction of a linear, homogeneous, isotropic, temporally dispersive, non-hysteretic medium filling the semi-infinite space $z \geq 0$ (where there are no charge or current sources) and is due to an input unit-amplitude, Gaussian pulse envelope modulated chirped harmonic wave [1, 2, 7, 11, 14], whose envelope has an arbitrary initial full width $2T$ centered around $t_0 > 0$, and it modulates a harmonic wave of applied carrier frequency ω_c that is linearly chirped with a constant, arbitrary chirp parameter C. In Eq. (1), the integration contour L may be the real frequency axis, or any other homotopic to this axis contour, in the complex ω plane; hereafter, the terms $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ denote the real and imaginary parts, respectively, of the complex quantity inside the curly brackets. Moreover, in Eq. (1) the frequency independent unified complex spectral amplitude $\tilde{U}_U$ is given by

$$\tilde{U}_U = \frac{2T\pi^{1/2}}{(4+i2CT^2)^{1/2}} \exp\left[-\left(\frac{t_0^2}{T^2} + i\psi\right)\right], \tag{2}$$

and the unified complex phase $\Phi_U(\omega,\theta)$ is given by

$$\Phi_U(\omega,\theta) = \phi(\omega,\theta) - \frac{cT^2\left(\omega - \omega_c - i\frac{2t_0}{T^2}\right)^2}{z\,(4+i2CT^2)}. \tag{3}$$

In Eq. (3), $\phi(\omega,\theta) = \{i\omega[n(\omega) - \theta]\}$ denotes the classical complex phase [3–5, 8, 13, 20], ψ is a constant phase term that is zero for a modulated chirped sine or $\pi/2$ for a cosine wave, $\theta = (ct)/z$ is a dimensionless space-time parameter and c is the vacuum speed of light. Throughout, the medium's refractive index $n(\omega)$ is described by the $(m+1)$ resonance Lorentz model

$$n(\omega) = \left[1 - \sum_{j=0}^{m} \frac{b_{2j}^2}{\omega^2 - \omega_{2j}^2 + i2\delta_{2j}\omega}\right]^{\frac{1}{2}}, \tag{4}$$

where, for the j-th resonance, $j = 0, 1, \ldots, m$, ω_{2j} is the resonance frequency, b_{2j}^2 is the square of the plasma frequency and δ_{2j} is the resonance linewidth. This causal model is often utilized [1–5, 8, 12, 13, 16] in pulse propagation problems when the field intensity is sufficiently low and the medium's nonlinearities are negligible.

2. UNIFIED ASYMPTOTIC DESCRIPTION OF CHIRPED GAUSSIAN PULSE PROPAGATION

The unified asymptotic description of the propagated field $A(z,t)$, which is obtained from an application of modern asymptotic techniques [13] on the integral expression (1), is given by

$$\mathrm{A}(z,t) = \sum_{l=1}^{n} \mathrm{A}_{Ul}(z,t), \quad n \leq 4(m+1)+1, \tag{5}$$

for all values of θ. Accordingly, $A(z,t)$ is composed of pulse components $A_{Ul}(z,t)$ each representing the asymptotic contribution of a respective relevant saddle point $\omega_{SP_{Ul}}(\theta)$ of $\Phi_U(\omega,\theta)$. In fact, $\Phi_U(\omega,\theta)$ has $n \leq [4(m+1)+1]$ saddle points which are considered to be isolated and of first-order for all values of θ.

When the first two terms in the asymptotic expansion are retained, each pulse component $\mathrm{A}_{Ul}(z,t)$ is given by

$$A_{Ul}(z,t) = \frac{1}{2\pi}\mathrm{Re}\left\{ i\exp\left\{\frac{z}{c}\Phi_U(\omega,\theta)\right\}\left(\frac{\pi c}{z}\right)^{\frac{1}{2}} \frac{\tilde{U}_U}{\left[-\frac{1}{2}\frac{d^2\Phi_U(\omega,\theta)}{d\omega^2}\right]^{\frac{1}{2}}}\left[\mathrm{H}(\omega,\theta) + O\left(z^{-\frac{5}{2}}\right)\right]\right\}_{\omega=\omega_{SP_{Ul}}(\theta)}, \tag{6}$$

where the quantity $H(\omega,\theta)$ is given by

$$\mathrm{H}(\omega,\theta) = 1 + \left(\frac{c}{2z}\right)\left\{\frac{\frac{15}{36}\left[\frac{d^3\Phi_U(\omega,\theta)}{d\omega^3}\right]^2 - \frac{1}{4}\left[\frac{d^4\Phi_U(\omega,\theta)}{d\omega^4}\right]\left[\frac{d^2\Phi_U(\omega,\theta)}{d\omega^2}\right]}{4\left[\frac{d^2\Phi_U(\omega,\theta)}{d\omega^2}\right]^2\left[-\frac{1}{2}\frac{d^2\Phi_U(\omega,\theta)}{d\omega^2}\right]^{\frac{3}{2}}}\right\}\left[-\frac{1}{2}\frac{d^2\Phi_U(\omega,\theta)}{d\omega^2}\right]^{\frac{1}{2}} \tag{7}$$

and is evaluated at the location of the respective relevant saddle point $\omega_{SP_{Ul}}(\theta)$, $l = 1, 2, \ldots, [4(m+1)+1]$. From Eq. (6) it then follows that the instantaneous angular frequency of oscillation $\omega_{IFO_{Ul}}(\theta)$ of the pulse component $A_{Ul}(z,t)$ is given by

$$\omega_{IFO_{Ul}}(\theta) = \mathrm{Re}\{\omega_{SP_{Ul}}(\theta)\} + \frac{1}{2}\frac{d}{dt}\left\{\arg\left[\frac{d^2\Phi_U(\omega_{SP_{Ul}}(\theta),\theta)}{d\omega^2}\right]\right\} - \frac{d}{dt}\{\arg[\mathrm{H}(\omega_{SP_{Ul}}(\theta),\theta)]\}. \tag{8}$$

Moreover, the envelope of $A_{Ul}(z,t)$ attains its stationary points (i.e., its local maxima and/or minima) when

$$\mathrm{Im}\{\omega_{SP_{Ul}}(\theta)\} = \frac{d}{dt}\left\{\ln\left[\frac{\left|-\frac{d^2\Phi_U(\omega_{SP_{Ul}}(\theta),\theta)}{d\omega^2}\right|^{\frac{1}{2}}}{|\mathrm{H}(\omega_{SP_{Ul}}(\theta),\theta)|}\right]\right\}. \tag{9}$$

3. DISCUSSION AND CONCLUSIONS

Hereafter, the input Gaussian pulse modulated chirped harmonic wave is chosen to have an envelope with initial width $2T = 3.891\,\mathrm{fs}$ centered around $t_0 = 15\,\mathrm{T}$, and it modulates a sine wave of applied carrier frequency $\omega_c = 1.615 \times 10^{15}\,\mathrm{s}^{-1}$ which is linearly chirped with a positive chirp parameter $C^+ = 0.028 \times 10^{30}\,\mathrm{s}^{-1}$. Moreover, the dynamical evolution of the propagated field is described at a propagation distance $z = 5.0z_d$, where $z_d = 14.392\,\mu\mathrm{m}$ is the absorption depth at ω_c, in the double resonance $(m = 1)$ fluoride-type glass of Refs. [4, 8]. Notice that, ω_c is chosen to be located at the inflection point of $n_r(\omega_r)$ and the input sine wave completes a single oscillation in the unchirped case. Furthermore, for the input positive chirp case, C^+ is such that ω_c doubles over the time interval $\Delta t = 30\,\mathrm{T}$.

The dynamics of the nine saddle points in the input positive chirp case are illustrated in Fig. 1. Moreover, Figs. 2(a) and 2(b) illustrate dynamical evolution of the (total) propagated field and the angular frequency of oscillation, respectively, evaluated using both the unified asymptotic approach as well as a numerical experiment. Notice that here, only the first term of the asymptotic expansion in Eq. (6) is retained. The prediction of the numerical experiment is in close agreement with the corresponding prediction of the unified asymptotic approach, even though no optimization has been applied. The propagated field $A(z,t)$ is mainly composed of the two dominant pulse components $A_{Ul}(z,t)$ $l = 1, 2$, each being due to the asymptotic contribution of a respective relevant saddle point $\omega_{SP_{Ul}}(\theta)$ of $\Phi_U(\omega,\theta)$. The peak amplitudes of the propagated field at $\theta^+_{UA_{\max}} = 1.381$ and at $\theta^+_{UB_{\max}} = 1.566$ occur, approximately, when the respective saddle points $\omega_{SP_{U2}}(\theta)$ and $\omega_{SP_{U1}}(\theta)$ intersect the real frequency axis at $\theta^+_{stnr_{U2,1}} = 1.376$ and at $\theta^+_{stnr_{U1,2}} = 1.548$, in agreement with Eq. (9).

Moreover, $\omega_{IFO_{Ul}}(\theta)$ is given by $\mathrm{Re}\{\omega_{SP_{Ul}}(\theta)\}$ as $\omega_{SP_{Ul}}(\theta)$ evolves with θ in the ω-plane; $\omega_{IFO_{Ul}}(\theta)$ lies in the real frequency range traversed by the $\mathrm{Re}\{\omega_{SP_{Ul}}(\theta)\}$ as θ increases, in agreement with Eq. (8).

The unified asymptotic approach provides an accurate description of the propagated field $A(z,t)$ in chirped Gaussian pulse propagation of arbitrary initial width in a multiple $(m+1)$ resonance, causally dispersive and absorptive Lorentz-type dielectric. The propagated field is a linear superposition of pulse components $A_{Ul}(z,t)$, $l \leq [4(m+1)+1]$ each being due to the asymptotic

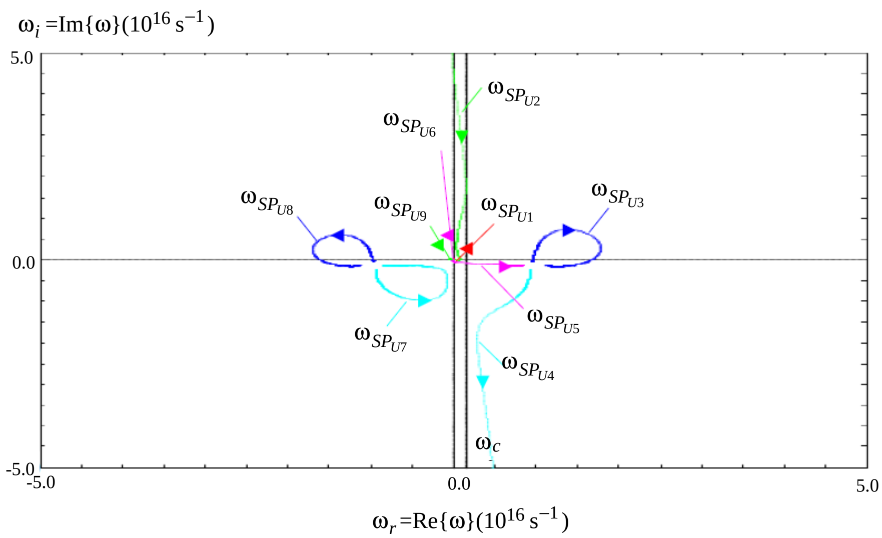
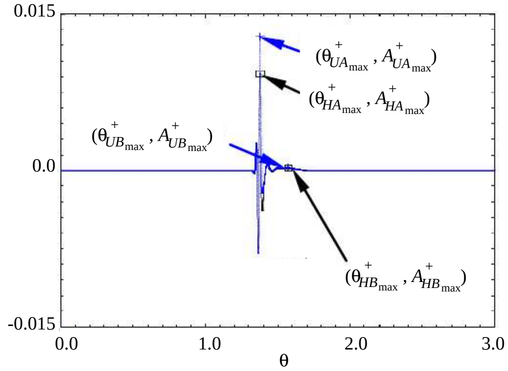
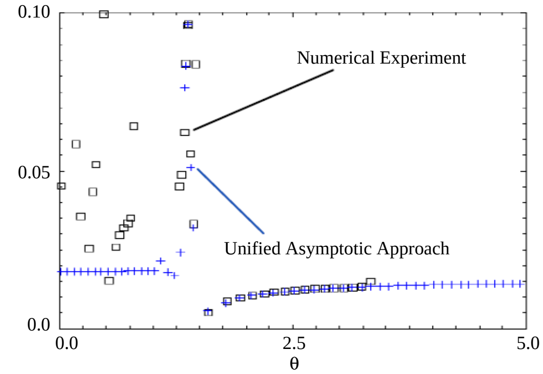


Figure 1: (Color online) dynamics of the saddle points $\omega_{SP_{Ul}}(\theta)$, $l = 1, 2, \ldots, 9$, of $\Phi_U(\omega,\theta)$ in the complex ω-plane, in the positive chirp case. The arrow at each path indicates the direction of motion of the respective saddle point as θ increases in the interval $\theta \in [-5.0, 15.0]$. When a saddle point crosses the real frequency axis, its coordinates are given by: $[\theta_{stnr_{U1,1}} = 0.211;\ \omega_{stnr_{U1,1}} = 0.018 \times 10^{16}\,\mathrm{s}^{-1}]$ and $[\theta_{stnr_{U1,2}} = 1.548;\ \omega_{stnr_{U1,2}} = 0.002 \times 10^{16}\,\mathrm{s}^{-1}]$ $[\theta_{stnr_{U2,1}} = 1.376;\ \omega_{stnr_{U2,1}} = 0.090 \times 10^{16}\,\mathrm{s}^{-1}]$ $[\theta_{stnr_{U3,1}} = -2.786;\ \omega_{stnr_{U3,1}} = 0.966 \times 10^{16}\,\mathrm{s}^{-1}]$ and $[\theta_{stnr_{U3,2}} = 1.329;\ \omega_{stnr_{U3,2}} = 1.679 \times 10^{16}\,\mathrm{s}^{-1}]$ $[\theta_{stnr_{U8,1}} = -2.786;\ \omega_{stnr_{U8,1}} = -0.967 \times 10^{16}\,\mathrm{s}^{-1}]$ and $[\theta_{stnr_{U8,2}} = 1.349;\ \omega_{stnr_{U8,2}} = -1.613 \times 10^{16}\,\mathrm{s}^{-1}]$ $[\theta_{stnr_{U9,1}} = 0.216;\ \omega_{stnr_{U9,1}} = -0.018 \times 10^{16}\,\mathrm{s}^{-1}]$ and $[\theta_{stnr_{U9,2}} = 1.409;\ \omega_{stnr_{U9,2}} = -0.042 \times 10^{16}\,\mathrm{s}^{-1}]$.



(a) Field Amplitude



(b) Oscillation Frequency ($10^{16}\,s^{-1}$)

Figure 2: (Color online) (a) the propagated field in the positive chirp case. The (black) squares ('□') denote the (absolute) maxima ($\theta^{+}_{HA_{\max}} = 1.382$, $A^{+}_{HA_{\max}} = 9.314 \times 10^{-3}$) and ($\theta^{+}_{HB_{\max}} = 1.575, A^{+}_{HB_{\max}} = 0.245 \times 10^{-3}$) of the field amplitude evaluated using the numerical experiment and denoted with the solid (black) line. The (blue) crosses ('+') denote the (absolute) maxima ($\theta^{+}_{UA_{\max}} = 1.381$, $A^{+}_{UA_{\max}} = 12.901 \times 10^{-3}$) and ($\theta^{+}_{UB_{\max}} = 1.566$, $A^{+}_{UB_{\max}} = 0.309 \times 10^{-3}$) of the field amplitude $A_U(z,t) = \sum_{l=1}^{m\leq 9} A_{Ul}(z,t)$ evaluated using the unified asymptotic approach and denoted with the solid (blue) line. (b) Evolution of the instantaneous angular frequencies of oscillation for the propagated fields depicted in (a). The (black) squares ('□') denote the instantaneous oscillation frequency evaluated using the numerical experiment. Moreover, the instantaneous oscillation frequency evaluated using the unified asymptotic approach is denoted with the (blue) cross ('+').

contribution of a relevant saddle point $\omega_{SP_{Ul}}(\theta)$ of the unified phase $\Phi_U(\omega,\theta)$ which appears in the exact integral expression of $A(z,t)$ and is a function of both the input field and the medium parameters as well as of the propagation distance. This asymptotic approach generalizes previous findings for Gaussian pulse propagation in a single resonance Lorentz medium which is valid and accurate from the delta function limit to the SVE pulse regime [3, 5, 13]. Moreover, this asymptotic approach describes the dynamical evolution of $\omega_{IFO_{Ul}}(\theta)$ (i.e., its value range, attained maxima and/or minima and the regions of up and down chirp), identifying the spectral components of the input pulse which contribute to the dynamical evolution of $A(z,t)$. The unified asymptotic approach provides unique insight on the interaction of the pulse components $A_{Ul}(z,t)$ which lead, in the sub-cycle pulse regime, to the break-up of $A(z,t)$ (for the considered double resonance Lorentzian, into a high frequency deformed Gaussian followed by a low frequency damped oscillatory pulse structure). The insight of the unified asymptotic description is useful in applications involving chirped pulses, especially in the extremely ultrashort pulse regime.

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