Block LU Preconditioner for the Electric Field Integral Equation

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Abstract— Boundary element discretizations are used in many fields of science and engineering, for example, to solve the problem of electromagnetic scattering with electric field integral equation (EFIE). To solve the EFIE on surfaces of arbitrary shape, RWG functions are traditionally used. The matrices arising from the discretization of integral equations can be adequately approximated with low-rank matrices using the mosaic-skeleton method. However, to solve the problem of scattering with a large wave size of the object, we have to solve a large system with $n \sim 10^5$–10^6. Systems of linear equations with matrices of this order are solved iteratively (GMRES). For a large wave size, the matrix is ill-conditioned and we have to use a preconditioner. In this paper, the number of iterations in the GMRES method is reduced by building a block LU preconditioner. The numerical experiments have revealed that the preconditioner makes it possible to solve the problem of electromagnetic scattering on objects with a larger wave size.

1. INTRODUCTION

The radiolocation problem is often solved using integral equations [1]. The simplest model when the given object is a perfectly conducting body is characterized by an integral equation of the electric field

$$\frac{1}{4\pi}\nabla_\tau \int\limits_S \nabla_N \cdot \bar{J}(N)\frac{e^{ik|M-N|}}{|M-N|}dS_N + \frac{k^2}{4\pi}\int\limits_S \bar{J}(N)\frac{e^{ik|M-N|}}{|M-N|}dS_N = \frac{ik}{\mu}\bar{E}_{0\tau}(M), \quad M \in S. \quad (1)$$

Equation (1) is numerically solved often by using RWG basis functions [2]. The problem is reduced to the solution of a system of linear algebraic equations with a dense matrix A. The size of this system (n) depends on the wave number k as well as on the shape and size of the surface S. The dependence on k has the form $n = \mathcal{O}(k^2)$; here, to obtain a solution of Equation (1) with an accuracy of $\varepsilon = 10^{-3}$ for an object with a typical size $a = 1\,\mathrm{m}$ for $k = 100\,\mathrm{m}^{-1}$, the system size n must be 10^4–10^5.

The solution of a system with a dense matrix of size 10^4–10^6 requires computing systems with a large (shared or distributed) RAM or special-purpose methods for compressing the dense matrix. For example, the studies [3, 4] used the Multilevel Fast Multipole Algorithm (MLFMA) and the mosaic-skeleton method to solve electrodynamic problems.

The mosaic-skeleton method [5] is more versatile than the MLFMA and can be easily parallelized, which makes the mosaic-skeleton method suitable for both electrodynamic problems and other applications [6].

The system with a dense matrix is solved by iterative methods that use only multiplication of a matrix by a vector and ignore the explicit construction of the inverse matrix. The projective iterative methods (for example, GMRES [7]) prove to be the most efficient. The computation of the radar cross section (RCS) of a perfectly conducting object requires that Equation (1) be solved with many right-hand sides. The number of right-hand sides is determined by the number of points on the scatter diagram and reaches a few thousands.

The system with several right-hand sides is solved by a modification of GMRES that uses basis vectors of the projective subspace for all right-hand sides of the system. Of course, the system could be solved for each right-hand side or the GMRES restarts could be used. However, in this case, the system solution time increases so greatly that it becomes impossible to solve the equation for $k > 10$.

A substantial drawback of the modified GMRES method is the large amount of RAM required for storing the basis vectors. The amount of memory is proportional to the matrix dimension n and number of iterations k_i. In the GMRES method, the number of iterations k_i grows with increasing of both n and wave number k. Therefore, for large wave numbers, the solution of the system requires considerable amount of RAM. The amount of this memory is much greater than that required for storing the compressed matrix.

To reduce the number of iterations and solve Equation (1) for large wave numbers, we propose here to build a preconditioner. The authors of [8] have already proposed a preconditioner for

Equation (1) on the basis of Calderon's formula. However, the approach proposed in [8] is suitable only for integral equations on closed surfaces. Our preconditioner can be efficiently used for both closed and open surfaces.

The papers [9, 10] also described preconditioners for compressed matrices in the $\mathcal{H}^2$ format. Like our preconditioner, these preconditioners contain parameters that should be chosen for a particular class of problems.

Unlike [9, 10], the preconditioner in this paper is constructed for a mosaic-skeleton rather than $\mathcal{H}^2$ matrix. In addition, our preconditioner is designed namely for electrodynamics problems rather than for integral equations that can be solved without preconditioners.

2. BLOCK LU PRECONDITIONER

Before constructing the preconditioner, we note that the inverse matrix that can be constructed from an LU decomposition is an ideal preconditioner. The solution of the system with a dense matrix using an LU decomposition requires $\mathcal{O}(n^3)$ operations. The solution with the help of the GMRES method requires $\mathcal{O}(n\log(n))$ operations. Nevertheless, for $n < 50000$, the system can be solved faster through an LU decomposition than using the GMRES algorithm. If the matrix is given in a compressed format, its approximate LU decomposition can be constructed in $o(n^3)$ operations. We use the approximation of L and U matrices as a preconditioner for solving the system by the iterative method.

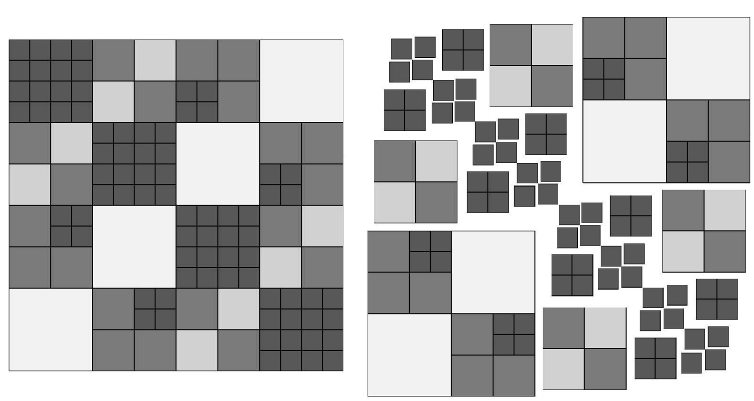
To construct the preconditioner, we assume that the non-degenerate square matrix A corresponding to integral Equation (1) is given in the mosaic-skeleton format [5]. In other words, the matrix A is divided into blocks, some of which are represented in the skeleton format, i.e., as $A_i = V_i W_i^T$, where for a block A_i of size $p\times q$, the matrices V_i and W_i have dimensions $p\times r$ and $q\times r$, respectively. The remaining blocks are dense and can be computed explicitly. An example of matrix division into blocks is shown in Figure 1(a). The compression degree of blocks is shown by grey-scale grading: the lighter a block, the more compressed it is. It follows from the mosaic-skeleton algorithm applied to the matrix obtained from the integral equation that the diagonal blocks are dense, square, and non-degenerate.

To construct a preconditioner, we mark the nondiagonal blocks in the matrix as shown in Figure 1(b). Next, we obtain a hierarchic block construct of the decomposition factors in accordance to the formula

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} \tag{2}$$

In (2), the blocks are computed in the following way:

1. $A_{11} = L_{11}D_1U_{11}$; we compute L_{11}, D_1, U_{11};
2. $A_{21} = L_{21}D_1U_{11}$; we use the formula $L_{21} = A_{21}U_{11}^{-1}D_1^{-1}$ to find L_{21};
3. $A_{12} = L_{11}D_1U_{12}$; we use the formula $U_{12} = D_1^{-1}L_{11}^{-1}A_{12}$ to find U_{12};
4 $A_{22} = L_{21}D_1U_{12} + L_{22}D_2U_{22}$; according to $L_{22}D_2U_{22} = A_{22} - L_{21}D_1U_{12}$ and using L_{21} and U_{12}, we find L_{22}, D_2, U_{22}.



(a) (b)

Figure 1: Mosaic division of the matrix.

In steps (2)–(4), the blocks of the matrices A, L and U are monolithic only at the initial stage. As can be seen from Figure 1(b), some parts of matrices may have a rather complex structure containing both dense and low-rank blocks. The L_{21} and U_{12} blocks cannot be computed as dense matrices because this operation is laborious and requires a considerable amount of RAM for large matrices. Therefore, the algorithm for constructing the block LU preconditioner uses the low-rank feature (i.e., the representation of original and computed blocks as VW^T).

In steps (2)–(4), as a result of the block summing operation, the ranks are summed; therefore, to ensure that the amount of data for storing the LU decomposition factors is small, the low-rank blocks are compressed up using singular value decomposition (SVD) with the truncation of a required number of lower singular numbers.

The singular numbers can be truncated using one of the following strategies:

1. fix the rank of the resulting block and assume that this rank does not exceed some value r_c;
2. fix the approximation accuracy ε_c of the compressed block.

In each case, we obtain an additional parameter that should be chosen for the given problem, i.e., for Equation (1). The value of this parameter affects the efficiency of the preconditioner. If the value of r_c is large or the accuracy ε_c is high, the construction of the preconditioner requires considerable computational resources. This means that in this case the preconditioner requires a long time to be constructed and takes much RAM. On the other hand, a small value of r_c or low accuracy ε_c can increase the number of iterations in the relevant algorithm.

An analysis of the preconditioner shows that it is difficult to choose the parameter r_c for the integral equation. For large values of r_c, the blocks mostly become dense but the number of iterations still remains large. It is more efficient to choose the approximation accuracy ε_c of low-rank approximation blocks.

When the matrix is represented as (2), the multiplication of the preconditioner by the vector is performed by $\mathcal{O}(n\log(n))$ operations.

3. NUMERICAL RESULTS

In the calculations, we investigate the scatter diagram: the dependence of the quantity

$$\sigma = \lim_{R\to\infty} 4\pi R \frac{\left|\bar{E}(R)\right|^2}{\left|\bar{E}_0\right|^2} \tag{3}$$

on the scattering angle φ of incident wave $\bar{E}_0$ for different angles.

Using $\bar{J}(N)$, we can write Formula (3) as

$$\sigma = 4\pi \left| \int_S (\bar{J}(N) - \bar{k}_0(\bar{k}_0 \cdot \bar{J}(N))k^2 e^{-ik|N|} dS_N \right|^2, \tag{4}$$

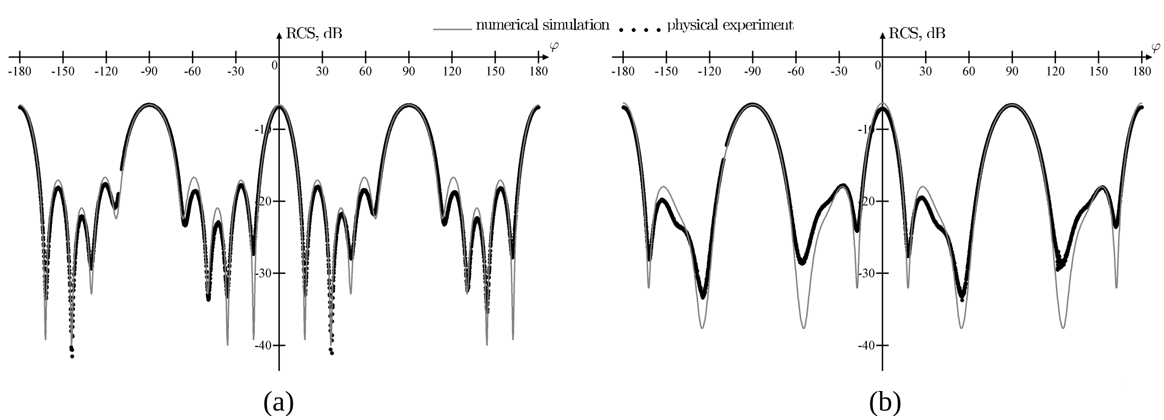


Figure 2: RCS for a cylinder with a frequency of 2 GHz ($k = 42\,\mathrm{m}^{-1}$); the matrix dimension is $n = 21760$. (a) Horizontal polarization. (b) Vertical polarization.

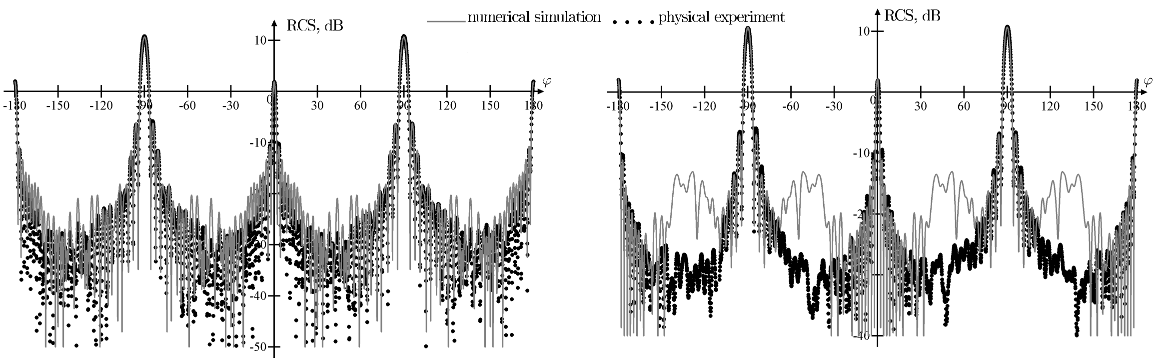
where $\bar{k}_0$ is the unit vector of the incident wave.

We consider a circular cylinder of diameter 0.15 m and length 0.25 m.

Figures 2–4 show the comparison of RCS calculations in decibels $10\log(\sigma/\sigma_0)$ with the results of experiments conducted by the Institute of Theoretical and Applied Electrodynamics of the Russian Academy of Sciences for two polarizations (horizontal and vertical).

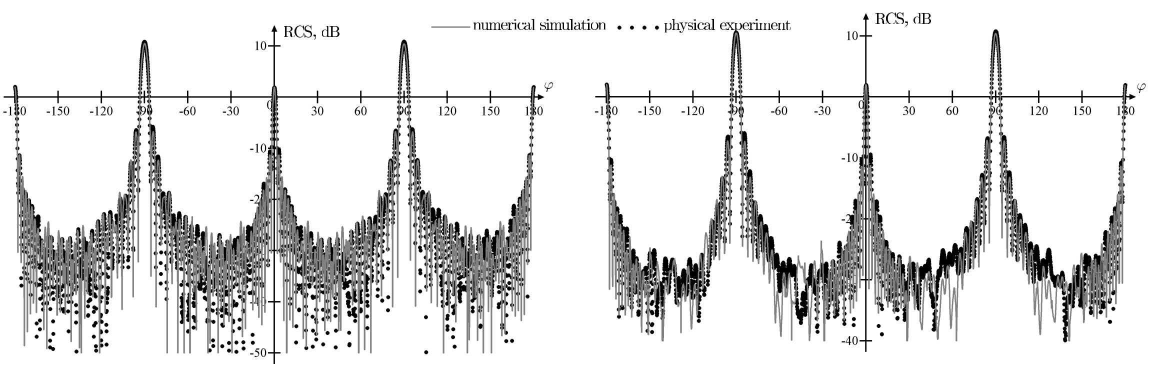
It can be seen from Figure 2 that for low frequencies there is a good agreement between calculated and experimental data for both horizontal and vertical polarizations. Figures 3 and 4 show that at high frequencies the elimination of false maxima in RCS requires a much finer grid.

Table 1 gives the following data:
k is the wave number (m^{-1}),
n is the number of unknown variables in the system,
T_1 is the matrix computation time (s),



(a) (b)

Figure 3: RCS for a cylinder with a frequency of 16 GHz ($k = 335\,\mathrm{m}^{-1}$), the matrix dimension is $n = 21760$. (a) Horizontal polarization. (b) Vertical polarization.



(a) (b)

Figure 4: RCS for a cylinder with a frequency of 16 GHz ($k = 335\,\mathrm{m}^{-1}$), the matrix dimension is $n = 192156$. (a) Horizontal polarization. (b) Vertical polarization.

Table 1: The results of calculations.

k	42	84	167	335
n	21 760	40068	90 063	192 156
T_1	1336	3 981	20 013	23 376
T_2	1210	5 336	-	-
T_3	549	1 297	4 216	12 425
T_4	42	124	550	1 167
M_0	3 613	12 249	61 885	281 707
M_1	961	3 300	6 893	15 946
M_2	3 083	7 883	-	-
M_3	1 102	4 125	9 788	22 495

T_2 is the solution time without the preconditioner (s),
T_3 is the preconditioner construction time (s),
T_4 is the solution time with the preconditioner (s),
M_0 is the memory taken by the matrix witout compression (Megabytes),
M_1 is the memory taken by the mosaic-skeleton matrix (Megabytes),
M_2 is the memory required for the solution of the system without the preconditioner (Megabytes),
M_3 is the memory taken by the preconditioner (Megabytes).

The calculations were conducted on a personal computer with 42 Gb of RAM for a cylinder with the parameters described above. The number of right-hand sides is 2048. The approximation accuracy and the solution accuracy were taken to be 10^{-3}. The preconditioner construction parameter ε_c was 10^{-4}.

Without the preconditioner, we managed to obtain the solution for $k = 42$ and $k = 84$. For $k = 167$ and $k = 335$, we managed to obtain the solution only with the preconditioner. Thus, the block LU preconditioner proposed in this study makes it possible to increase the solution frequency in 4 times.

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