Algebraic Regularization of Universal Functions in EM via Self-induced Hadamard Finite Parts

A. R. Baghai-Wadji
University of Cape Town, South Africa

Abstract— In boundary element method applications there are contradicting requirements which need to be accommodated skillfully in order to achieve high accuracy and high speed at the same time. Regarding acceleration of computations, it is desirable, besides parallelization and other conventional measures, to extract costly "universal" features from the computations which are common to a certain class of problems. If successful the resulting Universal Functions (UFs) can be pre-calculated and stored and referred to whenever necessary. As can be shown the introduction of UFs, however, requires additive factorization of terms, leading to high-order algebraic singularity in infrared region in spectral domain (zero and small values of the wavenumbers). Thereby, the algebraic order of the additional singularity depends on the degree of smoothness of the basis- and testing functions employed — the smoother the basis and testing functions the higher the order of singularity at the origin of the coordinate system in the spectral domain. On the other hand smooth basis- and weighting functions improve the convergence of the involved Fourier-type integrals algebraically in ultraviolet region (large values of the wavenumber in spectral domain). The main result in this contribution is the fact that all the aforementioned seemingly contradicting requirements can be reconciled naturally. The magic is done by recognizing that Hadamard Finite Parts appear in the introduced Universal Functions automatically, with one additional pleasant surprise: Ordinarily Hadamard Finite Parts technique, as the name implies, considers finite parts of infinite integrals, thus throwing away the infinities in the calculations. In the proposed procedure the infinities add up to zero exactly. Finally it should be pointed out that the Hadamard Finite Parts in the UFs are exclusively induced by the chosen basis- and testing functions, thus, justifying the using of the "self-induced" property in the title.

1. INTRODUCTION

In an accompanying paper a second renormalization technique has been proposed which ensures exponential decay of dyadic Green's functions in ultraviolet region in spectral domain. The exponential regularization has its genesis in the construction of problem-specific Dirac delta functions; it tempers the slowly-decaying divergent behavior of Green's functions. In contrast the algebraic regularization, proposed in this contribution, has its origin in the smoothness properties of the chosen basis- and weighting functions. Consequently, the next logical question is whether or not the two regularization techniques, exponential and algebraic, can be combined to even further enhance the quality of our computations. The answer is unconditionally affirmative. Thus the combined exponential- and algebraic regularization techniques promise ultra-precise calculations of the near-fields in EM simulations, while permitting the construction of Universal Functions. Furthermore, it is worth emphasizing that both regularization techniques apply to closed-form Green's functions as well Green's functions which can only be calculated numerically. The latter property is crucially important: it enables the simulation of nanoscale devices involving complex media.

2. PREPARATORY CONSIDERATIONS

Consider the following convolution type integral in spectral domain:

$$\varphi(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\bar{\varphi}(k_1,k_2)e^{jk_1x}e^{jk_2y}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\bar{G}(k_1,k_2)\bar{\rho}(k_1,k_2)e^{jk_1x}e^{jk_2y} \tag{1}$$

A bar indicates quantities in $(k_1,\, k_2)$-spectral domain. Furthermore, $\bar{G}\,(k_1,\, k_2)$ stands for a generic Green's function. It has been assumed that $G(x,y)$ is translationally invariant; i.e., G $(x\text{-}x',\, y\text{-}y')$, where $(x,\, y)$ and $(x',\, y')$, respectively, refer to the observation- and source point. In addition we shall assume the following source distribution with *a priori* unknown coefficients ρ_n:

$$\rho(x,y)=\sum_{n=1}^{N}\rho_n b_n(x,y)\quad\text{with}\quad b_n(x,y)=\begin{cases}1 & x_n^b<x<x_n^e\ \text{ and }\ y_n^b<y<y_n^e\\ 0 & \text{elsewhere}\end{cases} \tag{2}$$

The Fourier transform of $\rho(x, y)$ denoted by $\bar{\rho}(k_1, k_2)$ is:

$$\bar{\rho}(k_1,k_2)=\sum_{n=1}^{N}\rho_n\underbrace{\frac{e^{-jk_1x_n^e}-e^{-jk_1x_n^b}}{-jk_1}}_{2\triangle x_n\ \text{for}\ k_1\to 0}\underbrace{\frac{e^{-jk_2y_n^e}-e^{-jk_2y_n^b}}{-jk_2}}_{2\triangle y_n\ \text{for}\ k_2\to 0} \tag{3}$$

where $2\triangle x_n(=x_n^e-x_n^b)$ and $2\triangle y_n(=y_n^e-y_n^b)$, respectively, denote linear dimensions in the x- and y-direction of the nth sub-square. Denoting the integral of the source on the nth sub-square by Q_n we have $Q_n=\rho_n(2\triangle x_n)(2\triangle y_n)$. Using Q_n and rearranging (3) reads:

$$\bar{\rho}(k_1,k_2)=\sum_{n=1}^{N}\frac{Q_n}{-4\triangle x_n\triangle y_nk_1k_2}\left(e^{-jk_1x_n^e}-e^{-jk_1x_n^b}\right)\left(e^{-jk_2y_n^e}-e^{-jk_2y_n^b}\right) \tag{4}$$

Testing $\varphi(x,y)$ by the weighting functions $b_m(x,y)$, $m=1,\ldots,M$, and denoting the average of $\varphi(x,y)$ on the mth sub-sqaure by φ_m we have:

$$\varphi_m=\frac{1}{(x_m^e-x_m^b)(y_m^e-y_m^b)}\int_{x_m^b}^{x_m^e}\int_{y_m^b}^{y_m^e}dxdy\varphi(x,y) \tag{5}$$

Substituting the far-most term in (1) into (5) and rearranging the order of integrals we obtain

$$\varphi_m=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\bar{G}(k_1,k_2)\bar{\rho}(k_1,k_2)\underbrace{\frac{1}{4\triangle x_m\triangle y_m}\int_{x_m^b}^{x_m^e}dxe^{jk_1x}\int_{y_m^b}^{y_m^e}dye^{jk_2y}}_{\bar{w}_m(k_1,k_2)} \tag{6}$$

where we have introduced $\bar{w}_m(k_1,k_2)$ with

$$\bar{w}_m(k_1,k_2)=\frac{1}{4\triangle x_m\triangle y_m}\underbrace{\frac{e^{jk_1x_m^e}-e^{jk_1x_n^b}}{jk_1}}_{2\triangle x_m\ \text{for}\ k_1\to 0}\underbrace{\frac{e^{jk_2y_m^e}-e^{jk_1y_n^b}}{jk_2}}_{2\triangle y_m\ \text{for}\ k_2\to 0} \tag{7}$$

and $\lim_{(k_1,k_2)\to(0,0)}\bar{w}_m(k_1,k_2)=1$. In the following we shall use the form:

$$\bar{w}_m(k_1,k_2)=\frac{1}{-4\triangle x_m\triangle y_mk_1k_2}\left[e^{jk_1x_m^e}-e^{jk_1x_n^b}\right]\left[e^{jk_2y_m^e}-e^{jk_1y_n^b}\right];\quad (m=1,\ldots,M) \tag{8}$$

Considering (6) and introducing $\bar{R}_m(k_1,k_2)$ we have:

$$\varphi_m=\int_{\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\bar{G}(k_1,k_2)\underbrace{\bar{w}_m(k_1,k_2)\bar{\rho}(k_1,k_2)}_{\bar{R}_m(k_1,k_2)};\quad (m=1,\ldots,M) \tag{9}$$

The solvability condition requires $M=N$: choosing the same number of basis- and testing functions. Furthermore, note that the following properties are valid, as established above:

$$\lim_{(k_1,k_2)\to(0,0)}\bar{\rho}(k_1,k_2)=\sum_{n=1}^{N}Q_n,\quad \text{and}\quad \lim_{(k_1,k_2)\to(0,0)}\bar{w}_m(k_1,k_2)=1 \tag{10}$$

3. AUTOMATIC EMERGENCE OF HADAMARD FINITE PARTS

Consider $\bar{R}_m(k_1,k_2)$ as introduced in (9). It is evident that $\bar{R}_m(k_1,k_2)$ only depends on the selected basis- and testing functions, implying that the Green's function plays no role in the structure of $\bar{R}_m(k_1,k_2)$. On the other hand $\bar{R}_m(k_1,k_2)$ shall play a significant role in the algebraic regularization of the field integrals, to any degree desirable. Furthermore, $\bar{R}_m(k_1,k_2)$ will automatically give rise to the emergence of Hadamard Finite Parts ([1, Ref. 3]). In order to establish these results we substitute for $\bar{\rho}(k_1, k_2)$ and $\bar{w}_m(k_1, k_2)$, respectively, from (4) and (8), and rearrange to obtain:

$$\begin{aligned}\bar{R}_m(k_1,k_2) = &\sum_{n=1}^{N} Q_n \frac{1}{16\triangle x_m \triangle y_m \triangle x_n \triangle y_n k_1^2 k_2^2}\\ &\times\left[e^{j(k_1 x_m^e+k_2 y_m^e)} - e^{j(k_1 x_m^e+k_2 y_m^b)} - e^{j(k_1 x_m^b+k_2 y_m^e)} + e^{j(k_1 x_m^b+k_2 y_m^b)}\right]\\ &\times\left[e^{-j(k_1 x_n^e+k_2 y_n^e)} - e^{-j(k_1 x_n^e+k_2 y_n^b)} - e^{-j(k_1 x_n^b+k_2 y_n^e)} + e^{-j(k_1 x_n^b+k_2 y_n^b)}\right] \quad (11)\end{aligned}$$

Lemma 1 The following relationship holds true:

$$\begin{aligned}\bar{\mathcal{R}}_m(k_1,k_2) = &\sum_{n=1}^{N} Q_n \frac{1}{16\triangle x_m \triangle y_m \triangle x_n \triangle y_n k_1^2 k_2^2}\\ &\times\Big[+\Big\{1+[j\,(k_1 x_m^e + k_2 y_m^e)] + \frac{1}{2}[j(k_1 x_m^e + k_2 y_m^e)]^2\Big\}\\ &-\Big\{1+[j(k_1 x_m^e + k_2 y_m^b)] + \frac{1}{2}[j(k_1 x_m^e + k_2 y_m^b)]^2\Big\}\\ &-\Big\{1+[j(k_1 x_m^b + k_2 y_m^e)] + \frac{1}{2}[j(k_1 x_m^b + k_2 y_m^e)]^2\Big\}\\ &+\Big\{1+[j(k_1 x_m^b + k_2 y_m^b)] + \frac{1}{2}[j(k_1 x_m^b + k_2 y_m^b)]^2\Big\}\Big]\\ &\times\Big[+\Big\{1+[-j(k_1 x_n^e + k_2 y_n^e)] + \frac{1}{2}[-j(k_1 x_n^e + k_2 y_n^e)]^2\Big\}\\ &-\Big\{1+[-j(k_1 x_n^e + k_2 y_n^b)] + \frac{1}{2}[-j(k_1 x_n^e + k_2 y_n^b)]^2\Big\}\\ &-\Big\{1+[-j(k_1 x_n^b + k_2 y_n^e)] + \frac{1}{2}[-j(k_1 x_n^b + k_2 y_n^e)]^2\Big\}\\ &+\Big\{1+[-j(k_1 x_n^b + k_2 y_n^b)] + \frac{1}{2}[-j(k_1 x_n^b + k_2 y_n^b)]^2\Big\}\Big] = \sum_{n=1}^{N} Q_n \quad (12)\end{aligned}$$

Proof: A comparison between (11) and (12) reveals that the expression for $\bar{\mathcal{R}}_m\,(k_1,\,k_2)$ has been derived $\bar{R}_m\,(k_1,\,k_2)$ by replacing the exponential functions with their corresponding Taylor series expansions, and retaining the first three dominant terms only. Thus, we can expect that properties of $\bar{\mathcal{R}}_m\,(k_1,\,k_2)$ reveals the properties of $\bar{R}_m\,(k_1,\,k_2)$ in the limit $(k_1, k_2) \to (0, 0)$. The first pair of square brackets involve m-dependent x- and y coordinates of the mth testing function, while the second pair of square brackets involve n-dependent x- and y coordinates of the nth basis function. Focusing on the first m-dependent square brackets it is easily seen that the four ones cancel out. Furthermore, the four first-order terms taken collectively vanish. However, the quadratic terms result in $-k_1k_2(2\triangle x_m)(2\triangle y_m)$: *The quadratic terms do not cancel out.* Considering the second n-dependent square brackets analogous statements can be made *mutatis mutandis*: in particular the fact that quadratic terms sum up to $-k_1k_2(2\triangle x_n)(2\triangle y_n)$. Thus, multiplying the two square brackets gives $16\triangle x_m \triangle y_m \triangle x_n \triangle y_n k_1^2 k_2^2$ which exactly equals the denominator of the fraction standing after Q_n in the first line of (12); the latter fact establishes the proof.

Note that for systems with certain structural symmetry we may have $\sum_{n=1}^{N} Q_n = 0$; thus leading to $\bar{R}_m(k_1, k_2) = 0$. The above result suggests the following Lemma.

Lemma 2: The expression at the right-hand side of (11) is equivalent to the expression at the right-hand side of (13) given below, for arbitrary finite value of the constant α. (The cases $\alpha = 0, 1$ will be of particular interest for the purposes in this paper.)

$$\bar{R}_m\,(k_1,k_2)=\sum_{n=1}^{N} Q_n \frac{1}{16\triangle x_m \triangle y_m \triangle x_n \triangle y_n k_1^2 k_2^2}$$

$$
\begin{aligned}
&\times\Big\{+\exp\left\{j\left[k_1(x_m^e-x_n^e)+k_2(y_m^e-y_n^e)\right]\right\}-\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1(x_m^e-x_n^e)+k_2(y_m^e-y_n^e)\right]\right)^l\\
&-\exp\left\{j\left[k_1(x_m^e-x_n^e)+k_2\left(y_m^e-y_n^b\right)\right]\right\}+\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1(x_m^e-x_n^e)+k_2\left(y_m^e-y_n^b\right)\right]\right)^l\\
&-\exp\left\{j\left[k_1\left(x_m^e-x_n^b\right)+k_2(y_m^e-y_n^e)\right]\right\}+\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1\left(x_m^e-x_n^b\right)+k_2(y_m^e-y_n^e)\right]\right)^l\\
&+\exp\left\{j\left[k_1\left(x_m^e-x_n^b\right)+k_2\left(y_m^e-y_n^b\right)\right]\right\}-\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1\left(x_m^e-x_n^b\right)+k_2\left(y_m^e-y_n^b\right)\right]\right)^l\\
&-\exp\left\{j\left[k_1(x_m^e-x_n^e)+k_2\left(y_m^b-y_n^e\right)\right]\right\}+\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1(x_m^e-x_n^e)+k_2\left(y_m^b-y_n^e\right)\right]\right)^l\\
&+\exp\left\{j\left[k_1(x_m^e-x_n^e)+k_2\left(y_m^b-y_n^b\right)\right]\right\}-\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1(x_m^e-x_n^e)+k_2\left(y_m^b-y_n^b\right)\right]\right)^l\\
&+\exp\left\{j\left[k_1\left(x_m^e-x_n^b\right)+k_2\left(y_m^b-y_n^e\right)\right]\right\}-\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1\left(x_m^e-x_n^b\right)+k_2\left(y_m^b-y_n^e\right)\right]\right)^l\\
&-\exp\left\{j\left[k_1\left(x_m^e-x_n^b\right)+k_2\left(y_m^b-y_n^b\right)\right]\right\}+\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1\left(x_m^e-x_n^b\right)+k_2\left(y_m^b-y_n^b\right)\right]\right)^l\\
&-\exp\left\{j\left[k_1\left(x_m^b-x_n^e\right)+k_2(y_m^e-y_n^e)\right]\right\}+\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1\left(x_m^b-x_n^e\right)+k_2(y_m^e-y_n^e)\right]\right)^l\\
&+\exp\left\{j\left[k_1\left(x_m^b-x_n^e\right)+k_2\left(y_m^e-y_n^b\right)\right]\right\}-\alpha\sum_{l=0}^{3}\frac{1}{l!}\left\{j\left[k_1\left(x_m^b-x_n^e\right)+k_2\left(y_m^e-y_n^b\right)\right]\right)^l\\
&+\exp\left\{j\left[k_1\left(x_m^b-x_n^b\right)+k_2(y_m^e-y_n^e)\right]\right\}-\alpha\sum_{l=0}^{3}\frac{1}{l!}\left\{j\left[k_1\left(x_m^b-x_n^b\right)+k_2(y_m^e-y_n^e)\right]\right)^l\\
&-\exp\left\{j\left[k_1\left(x_m^b-x_n^b\right)+k_2\left(y_m^e-y_n^b\right)\right]\right\}+\alpha\sum_{l=0}^{3}\frac{1}{l!}\left\{j\left[k_1\left(x_m^b-x_n^b\right)+k_2\left(y_m^e-y_n^b\right)\right]\right)^l\\
&+\exp\left\{j\left[k_1\left(x_m^b-x_n^e\right)+k_2\left(y_m^b-y_n^e\right)\right]\right\}-\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1\left(x_m^b-x_n^e\right)+k_2\left(y_m^b-y_n^e\right)\right]\right)^l\\
&-\exp\left\{j\left[k_1\left(x_m^b-x_n^e\right)+k_2\left(y_m^b-y_n^b\right)\right]\right\}+\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1\left(x_m^b-x_n^e\right)+k_2\left(y_m^b-y_n^b\right)\right]\right)^l\\
&-\exp\left\{j\left[k_1\left(x_m^b-x_n^b\right)+k_2\left(y_m^b-y_n^e\right)\right]\right\}+\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1\left(x_m^b-x_n^b\right)+k_2\left(y_m^b-y_n^e\right)\right]\right)^l\\
&+\exp\left\{j\left[k_1\left(x_m^b-x_n^b\right)+k_2\left(y_m^b-y_n^b\right)\right]\right\}-\alpha\sum_{l=0}^{3}\frac{1}{l!}\left(j\left[k_1\left(x_m^b-x_n^b\right)+k_2\left(y_m^b-y_n^b\right)\right]\right)^l\Big\}
\end{aligned}
\tag{13}
$$

Proof: It is advantageous to introduce the following abbreviations: $x_{mn}^{ee}=x_m^e-x_n^e$, $x_{mn}^{eb}=x_m^e-x_n^b$, $x_{mn}^{be}=x_m^b-x_n^e$, and $x_{mn}^{bb}=x_m^b-x_n^b$. Similarly, introduce $y_{mn}^{ee}=y_m^e-y_n^e$, $y_{mn}^{eb}=y_m^e-y_n^b$, $y_{mn}^{be}=y_m^b-y_n^e$, and $y_{mn}^{bb}=y_m^b-y_n^b$. Next consider $\bar{\mathcal{R}}_m^{(3)}(k_1,k_2)$ involving the first three Taylor series expansion terms of the exponential functions in (13). Show that $\bar{\mathcal{R}}_m^{(3)}(k_1,k_2)=0$.

$$
\bar{\mathcal{R}}_m^{(3)}(k_1,k_2)=\sum_{n=1}^{N}Q_n\frac{1}{16\triangle x_m\triangle y_m\triangle x_n\triangle y_n k_1^2k_2^2}
$$

$$
\begin{aligned}
&\times\Big(+\Big\{1+j\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{ee}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{ee}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{ee}\Big]^3\Big\}\\
&-\Big\{1+j\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{eb}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{eb}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{eb}\Big]^3\Big\}\\
&-\Big\{1+j\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{ee}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{ee}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{ee}\Big]^3\Big\}\\
&+\Big\{1+j\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{eb}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{eb}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{eb}\Big]^3\Big\}\\
&-\Big\{1+j\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{be}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{be}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{be}\Big]^3\Big\}\\
&+\Big\{1+j\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{bb}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{bb}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{ee}+k_2y_{mn}^{bb}\Big]^3\Big\}\\
&+\Big\{1+j\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{be}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{be}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{be}\Big]^4\Big\}\\
&-\Big\{1+j\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{bb}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{bb}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{eb}+k_2y_{mn}^{bb}\Big]^3\Big\}\\
&-\Big\{1+j\Big[k_1x_{mn}^{be}+k_2y_{mn}^{ee}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{be}+k_2y_{mn}^{ee}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{be}+k_2y_{mn}^{ee}\Big]^3\Big\}\\
&+\Big\{1+j\Big[k_1x_{mn}^{be}+k_2y_{mn}^{eb}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{be}+k_2y_{mn}^{eb}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{be}+k_2y_{mn}^{eb}\Big]^3\Big\}\\
&+\Big\{1+j\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{ee}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{ee}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{ee}\Big]^3\Big\}\\
&-\Big\{1+j\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{eb}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{eb}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{eb}\Big]^3\Big\}\\
&+\Big\{1+j\Big[k_1x_{mn}^{be}+k_2y_{mn}^{be}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{be}+k_2y_{mn}^{be}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{be}+k_2y_{mn}^{be}\Big]^3\Big\}\\
&-\Big\{1+j\Big[k_1x_{mn}^{be}+k_2y_{mn}^{bb}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{be}+k_2y_{mn}^{bb}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{be}+k_2y_{mn}^{bb}\Big]^3\Big\}\\
&-\Big\{1+j\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{be}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{be}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{be}\Big]^3\Big\}\\
&+\Big\{1+j\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{bb}\Big]-\frac{1}{2}\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{bb}\Big]^2-\frac{j}{6}\Big[k_1x_{mn}^{bb}+k_2y_{mn}^{bb}\Big]^3\Big\}\Big)=0
\end{aligned}
\tag{14}
$$

In the 1st "column" the ± 1 terms cancel out pair-wisely. In the 2nd "column" the linear terms of the generic form $\pm j[k_1x_{mn}^{rs}+k_2y_{mn}^{pq}]$ add up to zero. In the 3rd "column" the quadratic terms cancel out collectively. In the 4th "column" the cubic terms add up to zero, completing the proof.

Had we included the quartic terms as well, they would have resulted in $\bar{\mathcal{R}}_m^{(4)}(k_1,k_2)=\sum_{n=1}^{N}Q_n$. Thus for systems with certain structural symmetry where $\sum_{n=1}^{N}Q_n=0$, even the 4th order terms would have canceled out. There is however, a fundamental difference between the terms up to and including the 3rd order terms, and terms higher than 3rd order: the columns corresponding to the 0th, 1st, 2nd, and 3rd order terms add up to zero column-wise and without involving any constraints upon the value of $\sum_{n=1}^{N}Q_n$, a fact which is not true for higher order terms. This distinction is the guiding principle for determining the number of terms in Taylor series expansions in Lemma 2.

Next consider the first "row" in (13). Adopt polar coordinates ($\bar{G}(k_1,k_2)=\bar{\mathcal{G}}(k,\theta)$) and subdivide the range of k into two sections $[0,\,k_c]$ and $[k_c,\,\infty)$. Here, k_c maybe any conveniently chosen or "critical" wavenumber. Furthermore, choose $\alpha=1$ when k varies in infrared region $[0,\,k_c]$ and $\alpha=0$ when k varies in ultraviolet domain $[k_c,\,\infty)$ leading to the following "Universal Function."

4. UNIVERSAL FUNCTIONS

Inspired by the aforementioned properties and Lemma 2 we define the geometry-independent "Universal Function," which can be pre-computed and stored for future numerical calculation:

$$
U(X,Y)=\int_0^{2\pi}d\theta\frac{1}{\sin^2\theta\cos^2\theta}\left\{\int_0^{k_c}dkk\bar{\mathcal{G}}(k,\theta)\frac{1}{k^4}\left(e^{jk(\sin\theta X+\cos Y)}-\sum_{l=0}^{3}\frac{1}{l!}\Big[jk(\sin\theta X+\cos Y)\Big]^l\right)\right.
$$

$$+\int_{k_c}^{\infty} dkk\bar{\mathcal{G}}(k,\theta)\frac{1}{k^4}e^{jk(\sin\theta X+\cos\theta Y)}\Bigg\} \tag{15}$$

The $\int_0^{k_c} dk$-integral in (15) resembles Hadamard Finite Part ([1, Ref. 3]). Note that for $k \ll 1$ the first dominant term in this integral is $o(k^4)$, which cancels the $1/k^4$-term perfectly. An important question is how to proceed if $\bar{\mathcal{G}}(k,\theta) \propto 1/|k|$ for $k \ll 1$. Then the total source in the universe must add up to zero ($\sum_{n=1}^{N} Q_n = 0$) (regularizing condition in the infrared region). In such a case we can include in (15) the 4th-order terms as well by letting l run from 0 to 4. On the other hand, note that the $\int_{k_c}^{\infty} dk$-integral in (15) decays sufficiently strongly for $k \to \infty$, thus ensuring convergence of the integral. Consequently, $U(X, Y)$ is regular in the infrared- as well as ultraviolet regions. In electrodynamics certain dyadic Green's functions are proportional to k for $k \to \infty$, implying that $k\bar{\mathcal{G}}(k,\theta)/k^4$ behaves according to $1/k^2$. Considering the $\int_{k_c}^{\infty} dk$-integral it can be concluded that the integrability in ultraviolet region is safely guaranteed even in such cases.

Upon construction it is immediate that the interaction elements in the Method of Moments applications, A_{mn}, can be written in terms of the Universal Function $U(X,Y)$ as follows:

$$\begin{aligned}A_{mn} = \frac{1}{64\pi^2\triangle x_m\triangle y_m\triangle x_n\triangle y_n}\Big\{&+U\left(x_{mn}^{ee}, y_{mn}^{ee}\right)-U\left(x_{mn}^{ee}, y_{mn}^{eb}\right)-U\left(x_{mn}^{eb}, y_{mn}^{ee}\right)+U\left(x_{mn}^{eb}, y_{mn}^{eb}\right)\\ &-U\left(x_{mn}^{ee}, y_{mn}^{be}\right)+U\left(x_{mn}^{ee}, y_{mn}^{bb}\right)+U\left(x_{mn}^{eb}, y_{mn}^{be}\right)-U\left(x_{mn}^{eb}, y_{mn}^{bb}\right)\\ &-U\left(x_{mn}^{be}, y_{mn}^{ee}\right)+U\left(x_{mn}^{be}, y_{mn}^{eb}\right)+U\left(x_{mn}^{bb}, y_{mn}^{ee}\right)-U\left(x_{mn}^{bb}, y_{mn}^{eb}\right)\\ &+U\left(x_{mn}^{be}, y_{mn}^{be}\right)-U\left(x_{mn}^{be}, y_{mn}^{bb}\right)-U\left(x_{mn}^{bb}, y_{mn}^{be}\right)+U\left(x_{mn}^{bb}, y_{mn}^{bb}\right)\Big\}\end{aligned} \tag{16}$$

It is worth noting that replacing $e^{jk(\sin\theta X+\cos\theta Y)}$ with $e^{jk(\sin\theta X+\cos\theta)Y}-1$ in the $\int_{k_c}^{\infty} dk$-integral, (15), does not alter the value of A_{mn}. Therefore, denoting the corresponding Universal Function by $V(X,Y)$, it can been shown that $V(0,0) = 0$, a result which is of theoretical and computational significance. For completeness it should be mentioned that MoM leads to $\sum_{n=1}^{N} A_{mn}Q_n = \varphi_m$.

5. CONCLUSION

A method has been proposed for algebraic regularization of slowly-decaying Fourier-type integrals arising in computational electromagnetics, when employing the Method of Moments (MoMs). It is shown that Hadamard Finite Part regularization scheme arises in the formulation automatically. The concept of Universal Functions for the calculation of the system matrix (self-action and mutual interaction) elements in MoMs applications, introduced previously, has been refined fundamentally. The results widen the scope of formulations proposed in [1] and the works cited therein significantly. The main result in this paper is that the proposed method does not impose any symmetry constraints on the dyadic Green's functions involved. A second important result is that the formalism permits o- and O-calculations by design; a property of theoretical and computational importance.

ACKNOWLEDGMENT

This work is based on the research supported in part by the National Research Foundation (UID: 93114). The support provided by University of Cape Town is also greatly appreciated.

REFERENCES

1. Baghai-Wadji, A. R., "3D electrostatic charge distribution on finitely-thick bus-bars in micro-acoustic devices: Combined regularization in the near- and far-field," *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, Special Issue, 2015.