Exponential Regularization of EM Dyadic Green's Functions via Green's Function-induced Dirac δ-functions

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Abstract— In a series of recently published contributions it was shown that singular or hypersingular dyadic Green's functions (GFs) in computational electrostatics and electromagnetics can be utilized to construct problem-specific integral representations of the Dirac δ-function, leading to parametrized smeared out δ_η-functions, which tend to the Dirac δ-function for the parameter η approaching 0^+. It was also shown that the constructed δ_η-functions can be employed to regularize originating GFs associated with Poisson equation in isotropic and anisotropic dielectrics. The main result of the present paper is that the constructed δ_η-function is used to regularize dyadic GFs associated with Maxwell's equations in isotropic media. The formulae arising in the formulation are all expressed in closed-form in spectral domain, allowing deep insights into the dynamics of the proposed regularization method. Complex media do not permit construction of GFs in closed form. Consequently, the integral representations for the δ_η-functions cannot be expressed in closed-form either. A recipe is proposed for numerically calculating δ_η-functions in asymptotic form. This result is claimed to present a genuine contribution to the computational physics. Applications in which near field phenomena play a role, e.g., nanoscopic and plasmonic devices will benefit from the results.

1. INTRODUCTION

Regularization problems arising in theoretical and computational field theory are fascinating and challenging at the same time; numerically, however, they may lead to catastrophically ill-conditioned results and are thus most undesirable. Singular fields, requiring regularization, have characteristic footprints in spectral domain. Electrostatic fields may possess a $1/|k|$ pole-like singularity in infrared wavenumber domain [1, 2]. Interestingly, the same $1/|k|$ functional dependence leads to slowly convergent or divergent Fourier-type integrals in ultraviolet wavenumber domain. The infrared $1/|k|$-singularity is associated with the fields in the far-field, while the $1/|k|$-ultraviolet slowly decaying, divergent behavior is related to the near-field in spatial domain, respectively. Singularities in electrodynamic fields manifest themselves in poles and branch points in intermediate regions in spectral domain. In computations the focus is on the regularization of slowly decaying dyadic Green's functions in spectral domain (near-field spatial domain). To this end the present author has proposed exponential — as well as algebraic regularization techniques. This paper is devoted to the exponential regularization which utilizes smeared out δ_η-functions constructed from the Green's functions themselves. (Complementary considerations in electrostatic limit have been treated in [1–5]. Static limits are illuminating and crucially important for their own right.)

2. CONVENTIONAL (SINGULAR) DYADIC GREEN'S FUNCTIONS IN 3D ELECTRODYNAMICS

Statement of the problem: Consider the dipoles $J_1\mathbf{e}_1\ \delta\ (x\text{-}x',\ y\text{-}y',\ z\text{-}z')$ and $J_2\mathbf{e}_2\ \delta\ (x\text{-}x',\ y\text{-}y',\ z\text{-}z')$ located at $(x',\ y',\ z')$ in a medium characterized by position-independent scalar permittivity ε and permeability μ. The unit vectors in the x- and y-directions are denoted by $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. Calculate the electric field vector $\mathbf{E}\ (x,\ y,\ z)$ and the magnetic field vector $\mathbf{H}\ (x,\ y,\ z)$, in entire $(x,\ y,\ z)$-space, as the response of the medium to the assumed unit dipoles.

Solution procedure: Partition $(x,\ y,\ z)$-space into subspaces $z > z'$ and $z < z'$ by introducing the fictitious plane $z = z'$. Find solution ansatzes for $\mathbf{E}(x,\ y,\ z)$ and $\mathbf{H}(x,\ y,\ z)$, involving *a priori* unknown coefficients, in regions $z > z'$ and $z < z'$. Satisfy Sommerfeld radiation conditions at infinity along with "interface" conditions at $z = z'$ to determine the unknown coefficients.

3. CALCULATION OF THE EIGENVALUES AND THE CORRESPONDING EIGENVECTORS

Maxwell's electrodynamic equations can be diagonalized with respect to, say, the z-coordinate, in closed form, which in spectral domain leads to a 4×4 algebraic eigenvalue equation with the following twofold degenerate eigenvectors ([5, Ref. [17]]):

$$\begin{bmatrix} E_1(k_1,k_2) \\ E_2(k_1,k_2) \\ H_1(k_1,k_2) \\ H_2(k_1,k_2) \end{bmatrix}^{\pm}_{(1)} = \begin{bmatrix} j\omega\mu\lambda^{\pm} \\ 0 \\ k_1k_2 \\ k_2^2-\varepsilon\mu\omega^2 \end{bmatrix}, \quad \begin{bmatrix} E_1(k_1,k_2) \\ E_2(k_1,k_2) \\ H_1(k_1,k_2) \\ H_2(k_1,k_2) \end{bmatrix}^{\pm}_{(2)} = \begin{bmatrix} 0 \\ -j\omega\mu\lambda^{\pm} \\ k_1^2-\varepsilon\mu\omega^2 \\ k_1k_2 \end{bmatrix} \tag{1}$$

corresponding to the eigenvalues $\lambda^{\pm} = \pm\sqrt{k_1^2+k_2^2-\varepsilon\mu\omega^2}$. Consequently we have four linearly independent eigenvectors. It is advantageous to introduce $W\,(k_1,\,k_2)$:

$$W(k_1,k_2) = \begin{cases} \sqrt{k_1^2+k_2^2-\varepsilon\mu\omega^2} & \text{if } k_1^2+k_2^2-\varepsilon\mu\omega^2>0 \\ -j\sqrt{\varepsilon\mu\omega^2-\left(k_1^2+k_2^2\right)} & \text{if } \varepsilon\mu\omega^2-\left(k_1^2+k_2^2\right)>0 \end{cases} \tag{2}$$

Since there is no danger of ambiguity same symbols are used to denote fields in spatial- and spectral domain, e.g., $E_1\,(x,\,y)$ and $E_1(k_1,k_2)$, rather than employing $E_1\,(x,\,y)$ and $\bar{E}_1\,(k_1,\,k_2)$.

Equipped with the above eigenpairs we now embark on constructing the dyadic Green's functions associated with the boundary value problem stated above, satisfying the integrability (Sommerfeld radiation) condition, and subject to the interface conditions which shall be formulated shortly.

3.1. General Field Expressions Satisfying Sommerfeld Radiation Condition

General field expressions satisfying Sommerfeld radiation condition in $z>z'$ and $z<z'$, respectively, take on the following forms:

$$\begin{aligned} \begin{bmatrix} E_1^{z>z'}(k_1,k_2) \\ E_2^{z>z'}(k_1,k_2) \\ H_1^{z>z'}(k_1,k_2) \\ H_2^{z>z'}(k_1,k_2) \end{bmatrix} &= A(k_1,k_2)\begin{bmatrix} -j\omega\mu W \\ 0 \\ k_1k_2 \\ k_2^2-\varepsilon\mu\omega^2 \end{bmatrix} e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W(z-z')} \\ &+B(k_1,k_2)\begin{bmatrix} 0 \\ j\omega\mu W \\ k_1^2-\varepsilon\mu\omega^2 \\ k_1k_2 \end{bmatrix} e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W(z-z')} \end{aligned} \tag{3a}$$

$$\begin{aligned} \begin{bmatrix} E_1^{z<z'}(k_1,k_2) \\ E_2^{z<z'}(k_1,k_2) \\ H_1^{z<z'}(k_1,k_2) \\ H_2^{z<z'}(k_1,k_2) \end{bmatrix} &= P(k_1,k_2)\begin{bmatrix} j\omega\mu W \\ 0 \\ k_1k_2 \\ k_2^2-\varepsilon\mu\omega^2 \end{bmatrix} e^{jk_1(x-x')}e^{jk_2(y-y')}e^{W(z-z')} \\ &+Q(k_1,k_2)\begin{bmatrix} 0 \\ -j\omega\mu W \\ k_1^2-\varepsilon\mu\omega^2 \\ k_1k_2 \end{bmatrix} e^{jk_1(x-x')}e^{jk_2(y-y')}e^{W(z-z')} \end{aligned} \tag{3b}$$

3.2. General Field Solutions at the Limits $z \to z'_+$ and $z \to z'_-$

In view of (3), and discarding the kernel $e^{jk_1(x-x')}e^{jk_2(y-y')}$, general field expressions at the limits $z \to z'_+$ and $z \to z'_-$, respectively, read:

$$\begin{bmatrix} E_1^{z>z'}(k_1,k_2) \\ E_2^{z>z'}(k_1,k_2) \\ H_1^{z>z'}(k_1,k_2) \\ H_2^{z>z'}(k_1,k_2) \end{bmatrix} = A(k_1,k_2)\begin{bmatrix} -j\omega\mu W \\ 0 \\ k_1k_2 \\ k_2^2-\varepsilon\mu\omega^2 \end{bmatrix} + B(k_1,k_2)\begin{bmatrix} 0 \\ j\omega\mu W \\ k_1^2-\varepsilon\mu\omega^2 \\ k_1k_2 \end{bmatrix} \tag{4a}$$

$$\begin{bmatrix} E_1^{z<z'}(k_1,k_2) \\ E_2^{z<z'}(k_1,k_2) \\ H_1^{z<z'}(k_1,k_2) \\ H_2^{z<z'}(k_1,k_2) \end{bmatrix} = P(k_1,k_2)\begin{bmatrix} j\omega\mu W \\ 0 \\ k_1k_2 \\ k_2^2-\varepsilon\mu\omega^2 \end{bmatrix} + Q(k_1,k_2)\begin{bmatrix} 0 \\ -j\omega\mu W \\ k_1^2-\varepsilon\mu\omega^2 \\ k_1k_2 \end{bmatrix} \tag{4b}$$

3.3. Interface Conditions (Discarding $e^{jk_1(x-x')}e^{jk_2(y-y')}$)

$$E_1^{z>z'}(k_1,k_2) - E_1^{z<z'}(k_1,k_2) = 0 \tag{5a}$$
$$E_2^{z>z'}(k_1,k_2) - E_2^{z<z'}(k_1,k_2) = 0 \tag{5b}$$
$$H_1^{z>z'}(k_1,k_2) - H_1^{z<z'}(k_1,k_2) = J_2 \tag{5c}$$
$$H_2^{z>z'}(k_1,k_2) - H_2^{z<z'}(k_1,k_2) = -J_1 \tag{5d}$$

Satisfying the interface conditions in (5) results in $P = -A$ and $Q = -B$ with:

$$A(k_1,k_2) = \frac{k_1^2 - \varepsilon\mu\omega^2}{2\varepsilon\mu\omega^2 W^2} J_1 + \frac{k_1 k_2}{2\varepsilon\mu\omega^2 W^2} J_2 \tag{6a}$$
$$B(k_1,k_2) = -\frac{k_1 k_2}{2\varepsilon\mu\omega^2 W^2} J_1 - \frac{k_2^2 - \varepsilon\mu\omega^2}{2\varepsilon\mu\omega^2 W^2} J_2 \tag{6b}$$

4. EXPLICIT EXPRESSIONS FOR THE GREEN'S FUNCTIONS

The Green's functions G_{E_1,J_1} and G_{E_1,J_2}: Having determined A, B, P and Q, and referring to (4) we obtain:

$$E_1^{z>z'}(k_1,k_2) = \left\{ \frac{j}{2\varepsilon\omega} \frac{k_2^2 - W^2}{W} \right\} J_1 + \left\{ \frac{j}{2\varepsilon\omega} \frac{-k_1 k_2}{W} \right\} J_2 \tag{7}$$

Continuity condition, Equation (5a), dictates the equality of the expressions for $E_1^{z<z'}$ and $E_1^{z>z'}$. Thus, considering the defining equation for the Green's functions G_{E_1,J_1} and G_{E_1,J_2}; i.e., $E_1 = \{G_{E_1,J_1}\} J_1 + \{G_{E_1,J_2}\} J_2$ we have:

$$G_{E_1,J_1}(k_1,k_2) = \frac{j}{2\varepsilon\omega} \frac{k_2^2 - W^2}{W} \quad \text{and} \quad G_{E_1,J_2}(k_1,k_2) = \frac{j}{2\varepsilon\omega} \frac{-k_1 k_2}{W} \tag{8}$$

The Green's functions G_{E_2,J_1} and G_{E_2,J_2}: Similarly, from $E_2 = \{G_{E_2,J_1}\}J_1 + \{G_{E_2,J_2}\}$ we obtain:

$$G_{E_2,J_1}(k_1,k_2) = \frac{j}{2\varepsilon\omega} \frac{-k_1 k_2}{W} \quad \text{and} \quad G_{E_2,J_2}(k_1,k_2) = \frac{j}{2\varepsilon\omega} \frac{k_1^2 - W^2}{W} \tag{9}$$

The Green's functions G_{H_1,J_1} and G_{H_1,J_2}: Substituting for A and B into the expressions for $H_1^{z>z'}$ and $H_1^{z<z'}$ we realize that the coefficients associated with J_1 cancel out, and the coefficient of J_2 equals $\varepsilon\mu\omega^2 W^2$ leading to:

$$H_1^{z>z'}(k_1,k_2) = \{0\} J_1 + \left\{ \frac{1}{2} \right\} J_2 \quad \text{and} \quad H_1^{z<z'}(k_1,k_2) = \{0\} J_1 + \left\{ -\frac{1}{2} \right\} J_2 \tag{10}$$

Introducing the sgn-function the two representations in (10) unify, resulting in:

$$H_1(k_1,k_2) = \{0\} J_1 + \left\{ \operatorname{sgn}(z-z') \frac{1}{2} \right\} J_2 \tag{11}$$

Thus from $H_1 = \{G_{H_1,J_1}\}J_1 + \{G_{H_1,J_2}\}J_2$ we have:

$$G_{H_1,J_1}(k_1,k_2) = 0 \quad \text{and} \quad G_{H_1,J_2}(k_1,k_2) = \operatorname{sgn}(z-z') \frac{1}{2} \tag{12}$$

The Green's functions G_{H_2,J_1} and G_{H_2,J_2}: Analogous considerations result in:

$$G_{H_2,J_1}(k_1,k_2) = -\operatorname{sgn}(z-z') \frac{1}{2} \quad \text{and} \quad G_{H_2,J_2}(k_1,k_2) = 0 \tag{13}$$

4.1. Field Distributions $E_3(\mathbf{x}, \mathbf{x}')$ and $H_3(\mathbf{x}, \mathbf{x}')$

For the components E_3 and H_3 normal to the interface ($z = z'$)-plane we employ the relationships

$$E_3(\mathbf{x}, \mathbf{x}') = -\frac{1}{j\omega\varepsilon}\left(-\partial_y H_1 + \partial_x H_2\right) \quad \text{and} \quad H_3(\mathbf{x}, \mathbf{x}') = \frac{1}{j\omega\mu}\left(-\partial_y E_1 + \partial_x E_2\right) \tag{14}$$

or, equivalently, in spectral domain:

$$E_3(k_1, k_2) = \frac{k_2}{\omega\varepsilon} H_1(k_1, k_2) - \frac{k_1}{\omega\varepsilon} H_2(k_1, k_2) \quad \text{and} \quad H_3(k_1, k_2) = -\frac{k_2}{\omega\mu} E_1(k_1, k_2) + \frac{k_1}{\omega\mu} E_2(k_1, k_2) \tag{15}$$

The Green's functions G_{E_3,J_1} and G_{E_3,J_2}: Substitute $H_1(k_1, k_2)$ and $H_2(k_1, k_2)$ into (15) to obtain:

$$G_{E_3,J_1}(k_1, k_2) = \mathrm{sgn}(z - z')\frac{k_1}{2\omega\varepsilon} \quad \text{and} \quad G_{E_3,J_2}(k_1, k_2) = \mathrm{sgn}(z - z')\frac{k_2}{2\omega\varepsilon} \tag{16}$$

The Green's functions G_{H_3,J_1} and G_{H_3,J_2}: Substitute $E_1(k_1, k_2)$ and $E_2(k_1, k_2)$ into (15) to obtain:

$$G_{H_3,J_1}(k_1, k_2) = -\frac{jk_2}{2W} \quad \text{and} \quad G_{H_3,J_2}(k_1, k_2) = \frac{jk_1}{2W} \tag{17}$$

5. PROBLEM-TAILORED FIELD-INDUCED EXPRESSION FOR DIRAC δ-FUNCTION

The results obtained so far allow us gaining a deeper understanding of the properties of Green's functions which are, as we shall see momentarily, of paramount significance for numerical calculations. In this section the construction of Dirac δ-function from the associated field distribution will be pursued in terms of one example; i.e., $G_{H_2,J_1}(\mathbf{x}|\mathbf{x}')$, which is the magnetic field component in y-direction in response to a dipole oriented along the x-axis, (refer to the left equation in (13)):

$$G_{H_2,J_1}\left(\mathbf{x}|\mathbf{x}'\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\left\{-\mathrm{sgn}(z - z')\frac{1}{2}\right\} e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W|z-z'|} \tag{18}$$

The term in the curly brackets is independent of the integration variables. Considering the definition of the function $\mathrm{sgn}(z - z')$ for regions $z > z'$ and $z < z'$, respectively, we obtain:

$$G_{H_2,J_1}^{z>z'}\left(\mathbf{x}|\mathbf{x}'\right) = -\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W|z-z'|} \tag{19a}$$

$$G_{H_2,J_1}^{z<z'}\left(\mathbf{x}|\mathbf{x}'\right) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W|z-z'|} \tag{19b}$$

The expressions in (19a) and (19b), respectively, stand for $H_2^{z>z'}$ and $H_2^{z<z'}$ as responses to a dipole oriented along the x-axis. In view of this fact we consider the following expression:

$$\begin{aligned}
&-\lim_{z\to z'_+} G_{H_2,J_1}^{z>z'}\left(\mathbf{x}|\mathbf{x}'\right) + \lim_{z\to z'_-} G_{H_2,J_1}^{z<z'}\left(\mathbf{x}|\mathbf{x}'\right) \\
&= \lim_{\eta\to 0^+}\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W\eta} + \lim_{\eta\to 0^+}\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W\eta} \\
&= \lim_{\eta\to 0^+}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W\eta}
\end{aligned} \tag{20}$$

On the other hand the interface condition for the field component H_2 reads:

$$-\lim_{z\to z'_+} G_{H_2,J_1}^{z>z'}\left(\mathbf{x}|\mathbf{x}'\right) + \lim_{z\to z'_-} G_{H_2,J_1}^{z<z'}\left(\mathbf{x}|\mathbf{x}'\right) = \delta(x - x', y - y') \tag{21}$$

A comparison between (20) and (21) results in:

$$\delta(x-x',y-y') = \lim_{\eta\to 0^+} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} e^{jk_1(x-x')} e^{jk_2(y-y')} e^{-W\eta} \tag{22}$$

Lemma: The double integra in (22) is a valid representation for the 2D Dirac δ-function [3].

The representation in (22) enables the regularization of singular integral expressions arising in the Method of Moments (MoM) applications. The integral in (22) is well-defined for any finite value $\eta > 0$, and can be utilized for defining the "smeared out" δ-function source, $\rho_\eta(x-x', y-y')$:

$$\delta_\eta(x-x',y-y') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} e^{jk_1(x-x')} e^{jk_2(y-y')} e^{-W\eta} \Longleftrightarrow e^{-W\eta} \quad \text{(in spectral domain)} \tag{23}$$

6. EXPONENTIALLY η-REGULARIZED DYADIC GREEN'S FUNCTIONS

Rather than exciting the medium with sources $\mathbf{J}_1 = J_1\mathbf{e}_1\delta$ $(x\text{-}x', y\text{-}y')$ and $\mathbf{J}_2 = J_2\mathbf{e}_2\delta$ $(x\text{-}x', y\text{-}y')$, consider their corresponding field-theoretically constructed "smeared out" versions $\mathbf{J}_{1,\eta} = J_1\mathbf{e}_1\delta_\eta$ $(x\text{-}x', y\text{-}y')$, and $\mathbf{J}_{2,\eta} = J_2\mathbf{e}_2\delta_\eta(x-x', y-y')$, respectively. Note that the support of $\mathbf{J}_{1,\eta}$ and $\mathbf{J}_{1,\eta}$ are confined to the interface. Using these problem-tailored distributed sources in the interface conditions we obtain the following exponentially η-regularized Green's functions:

$$\left\{\begin{matrix} G_{E_1,J_1,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{E_2,J_1,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_1,J_1,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_1,J_1,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{E_3,J_1,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_3,J_1,\eta}(\mathbf{x}|\mathbf{x}') \end{matrix}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} \left\{\begin{matrix} \frac{j}{2\varepsilon\omega}\frac{k_2^2-W^2}{W} \\ \frac{j}{2\varepsilon\omega}\frac{-k_1k_2}{W} \\ 0 \\ -\mathrm{sgn}(z-z')\frac{1}{2} \\ \mathrm{sgn}(z-z')\frac{k_1}{2\varepsilon\omega} \\ -\frac{jk_2}{2W} \end{matrix}\right\} e^{jk_1(x-x')} e^{jk_2(y-y')} e^{-W(|z-z'|+\eta)} \tag{24}$$

Similarly,

$$\left\{\begin{matrix} G_{E_1,J_2,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{E_2,J_2,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_1,J_2,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_1,J_2,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{E_3,J_2,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_3,J_2,\eta}(\mathbf{x}|\mathbf{x}') \end{matrix}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} \left\{\begin{matrix} \frac{j}{2\varepsilon\omega}\frac{-k_1k_2}{W} \\ \frac{j}{2\varepsilon\omega}\frac{k_1^2-W^2}{W} \\ \mathrm{sgn}(z-z')\frac{1}{2} \\ 0 \\ \mathrm{sgn}(z-z')\frac{k_2}{2\varepsilon\omega} \\ \frac{jk_1}{2W} \end{matrix}\right\} e^{jk_1(x-x')} e^{jk_2(y-y')} e^{-W(|z-z'|+\eta)} \tag{25}$$

Note that the original slowly-convergent or even divergent integrals have become regularized with a problem-specific exponential damping term. The subindex η is a reminder of this property!

7. EXPONENTIAL η-REGULARIZATION OF DYADIC GREEN'S FUNCTIONS IN COMPLEX MEDIA

Consider bi-anisotropic media characterized by material matrices $\underline{\underline{\varepsilon}}$, $\underline{\underline{\xi}}$, $\underline{\underline{\zeta}}$ and $\underline{\underline{\mu}}$, with constitutive equations $\mathbf{D} = \underline{\underline{\varepsilon}}\mathbf{E} + \underline{\underline{\xi}}\mathbf{H}$ and $\mathbf{B} = \underline{\underline{\zeta}}\mathbf{E} + \underline{\underline{\mu}}\mathbf{H}$. It can be shown that the following z-diagonalized and

(x, y)-supplementary representations are consistently equivalent with Maxwell's equations:

$$\left\{ \begin{bmatrix} \zeta_{21} & \zeta_{22} & \mu_{21} & \mu_{22} \\ -\zeta_{11} & -\zeta_{12} & -\mu_{11} & -\mu_{12} \\ -\varepsilon_{21} & -\varepsilon_{22} & -\xi_{21} & -\xi_{22} \\ \varepsilon_{11} & \varepsilon_{12} & \xi_{11} & \xi_{12} \end{bmatrix} + \begin{bmatrix} \zeta_{23} + \partial_x/j\omega & \mu_{23} \\ -\zeta_{13} + \partial_y/j\omega & -\mu_{13} \\ -\varepsilon_{23} & -\xi_{23} + \partial_x/j\omega \\ \varepsilon_{13} & \xi_{13} + \partial_y/j\omega \end{bmatrix} \begin{bmatrix} \varepsilon_{33} & \xi_{33} \\ \zeta_{33} & \mu_{33} \end{bmatrix}^{-1} \right.$$
$$\left. \times \begin{bmatrix} -\varepsilon_{31} & -\varepsilon_{32} & -\zeta_{31}+\partial_y/j\omega & -\zeta_{32}-\partial_x/j\omega \\ -\zeta_{31}-\partial_y/j\omega & -\zeta_{32}+\partial_x/j\omega & -\mu_{31} & -\mu_{32} \end{bmatrix} \right\} \begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} = \frac{\partial_z}{j\omega} \begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} \tag{26a}$$

$$\begin{bmatrix} E_3 \\ H_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_{33} & \xi_{33} \\ \zeta_{33} & \mu_{33} \end{bmatrix}^{-1} \begin{bmatrix} -\varepsilon_{31} & -\varepsilon_{32} & -\zeta_{31} + \frac{\partial_y}{j\omega} & -\zeta_{32} - \frac{\partial_x}{j\omega} \\ -\zeta_{31} - \frac{\partial_y}{j\omega} & -\zeta_{32} + \frac{\partial_x}{j\omega} & -\mu_{31} & -\mu_{32} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} \tag{26b}$$

Equation (26) in spectral domain can be used to construct asymptotic limits of the corresponding eigenpairs for $k \to 0$, and, particularly, for $k \to \infty$, to any order desired. Generally, the calculations need to be carried out numerically. The asymptotic limits of the eigenpairs allow the determination of the asymptotic limits of the Green's functions in spectral domain. Following the recipe introduced in this paper, "smeared out" parametrized δ-functions $\delta_\eta\,(x\text{-}x', y\text{-}y')$ can be constructed numerically. The latter functions can be employed as source functions to η-regularize Green's functions. The validity of this scheme has already been established theoretically. Current efforts concern the automatic- and symbolic implementation of the procedures involved.

8. CONCLUSIONS

A dyadic Green's functions-based smeared out δ-source function, $\delta_\eta\,(x\text{-}x', y\text{-}y')$ has been constructed and utilized to regularize originating Green's functions exponentially in ultraviolet region. The framework for applying the proposed method to complex media has also been provided.

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