Microwave Imaging of Dispersive Scatterers Using Vectorial Lagrange Multipliers

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Abstract— A time-domain microwave imaging method is proposed for the reconstruction of the spatial distribution of the electromagnetic properties of dispersive scatterers. The method is based on the minimization of a cost functional, which describes the discrepancy between measured and estimated scattered field data. The Maxwell's curl equations are introduced in an augmented cost functional, as equality constraints, via vectorial Lagrange multipliers. By means of the calculus of variations and the stationarity condition, it is shown that the vectorial Lagrange multipliers are the solution of the adjoint scattering problem. Moreover, the Fréchet derivatives of the cost functional with respect to the distributions of the scatterer properties are derived analytically. As a result, the Fréchet derivatives obtained can be utilized by any gradient-based optimization technique that updates the estimated scatterer properties iteratively. In this study, two-dimensional dispersive scatterers have been investigated. In the case of Debye scatterers, the spatial distributions of the relaxation time, the static and the optical permittivity, are reconstructed simultaneously. Also, the method is applied to the reconstruction of the resonant frequency, the damping factor, the static and the optical permittivity of Lorentz dispersive media.

1. INTRODUCTION

Inverse electromagnetic scattering problems have attracted significant research interest over the last years because of their applications in medical imaging, geophysical prospecting, nondestructive testing etc. [1]. The solution of an inverse scattering problem is difficult because it is an ill-posed and nonlinear problem [2]. Frequency-domain and the time-domain techniques are the two approaches for the solution of this problem, depending on the time variation of the excitations used. In the first case, the excitation is considered to be monochromatic and the inverse scattering problem is formulated in the frequency domain [3–5]. In single frequency-domain microwave imaging, the case of dispersive scatterers is tackled similarly to the case of nondispersive ones because monochromatic excitations do not generate dispersion phenomena. The second approach, namely the time-domain one, considers cases where the excitation field is wideband [6–8]. Time-domain microwave imaging results in improved reconstruction resolution of nondispersive scatterers, while it provides the ability to reconstruct dispersive scatterers. In particular, if the scatterer properties are frequency-dependent, dispersion phenomena appear, thus, time-domain techniques that estimate the parameters of dispersive media are important [9, 10].

In this study, a method to reconstruct simultaneously the electromagnetic properties of two-dimensional Debye and Lorentz dispersive scatterers is presented. The method minimizes the cost functional that describes the discrepancy between measured and estimated electric field values. By means of the calculus of variations, we derive the equations of the adjoint scattering problem. Furthermore, the Fréchet derivatives of the cost functional with respect to the medium properties are obtained. These derivatives are used by the Polak-Ribière optimization algorithm to estimate iteratively the scatterer properties while the finite-difference time-domain (FDTD) method is utilized for the solution of the direct and the adjoint scattering problem. As a result, it is possible to reconstruct the static and optical relative permittivity as well as the relaxation time of Debye scatterers, simultaneous. In the case of Lorentz dispersive scatterers, the static and optical relative permittivity as well as the resonant frequency and the damping factor are reconstructed. The numerical results that are presented, exhibit the efficiency of the proposed methodology.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

2.1. Direct Scattering

The relative complex permittivity of an inhomogeneous Debye scatterer is given by

$$\varepsilon_r(\omega, \vec{r}) = \varepsilon_\infty(\vec{r}) + \Delta\varepsilon(\vec{r}) / \left[1 + j\omega\tau(\vec{r})\right] \tag{1}$$

where ε_∞ is the optical relative permittivity, τ is the relaxation time, $\Delta\varepsilon = \varepsilon_s - \varepsilon_\infty$ (ε_s is the static relative permittivity), and ω is the angular frequency. In the case of a Lorentz dispersive scatterer, the relative permittivity is given by

$$\varepsilon_r(\omega, \vec{r}) = \varepsilon_\infty(\vec{r}) + \Delta\varepsilon(\vec{r})\omega_0^2(\vec{r})/\left[\omega_0^2(\vec{r}) + 2j\omega\zeta(\vec{r}) - \omega^2\right] \tag{2}$$

where ω_0 is the resonant frequency, and ζ is the damping factor.

The Maxwell's curl equations governing the wave propagation in a dispersive medium involve the polarization current density $\vec{J}$, i.e.,

$$\nabla \times \vec{E} + \mu\partial_t\vec{H} = 0, \quad \nabla \times \vec{H} - \varepsilon_0\varepsilon_\infty\partial_t\vec{E} - \vec{J} - \vec{J}_s = 0 \tag{3}$$

where $\vec{J}_s$ is the excitation current density, while $\vec{J}$ satisfies the differential equation

$$\vec{J} + \tau\partial_t\vec{J} - \varepsilon_0\Delta\varepsilon\partial_t\vec{E} = 0 \quad \text{or} \quad \omega_0^2\vec{J} + 2\zeta\partial_t\vec{J} + \partial_t^2\vec{J} - \varepsilon_0\Delta\varepsilon\omega_0^2\partial_t\vec{E} = 0 \tag{4}$$

in Debye or Lorentz scatterers, respectively.

2.2. Inverse Scattering

We assume that the dispersive scatterer is nonmagnetic ($\mu = \mu_0$) and lies within the scatterer domain D (see Fig. 1). The domain D is excited by I incident waves, while for each incidence the electric field is measured at K positions around the scatterer for the time interval $[0, T]$. Hence, a set of $I \times K$ electric field measurements is obtained, which are denoted as $\vec{E}_{ik}^m$ where $i = 1, \ldots, I$ and $k = 1, \ldots, K$. We note that the time of measurement T is selected in a way that the measured field at the farthest receiver has significantly faded out. The reconstruction of the scatterer properties (denoted by the vector $\mathbf{p}$) is based on the minimization of a cost functional, which describes the difference between measured and estimated values of the electric field. The Maxwell's curl equations and the polarization relation are introduced in the augmented cost functional as equality constraints by means of vectorial Lagrange multipliers. In particular the augmented cost functional is given by

$$F(\mathbf{p}) = \frac{1}{2}\sum_{i=1}^{I}\sum_{k=1}^{K}\int_0^T \left\|\vec{E}_{ik} - \vec{E}_{ik}^m\right\|^2 dt + MC + PC \tag{5}$$

where MC is the term related to the Maxwell's equations equality constraints, i.e.,

$$MC = \sum_{i=1}^{I}\int_0^T\int_V \left[\vec{h}_i \cdot \left(\nabla \times \vec{E}_i + \mu\partial_t\vec{H}_i\right) + \vec{e}_i \cdot \left(\nabla \times \vec{H}_i - \varepsilon_0\varepsilon_\infty\partial_t\vec{E}_i - \vec{J}_i - \vec{J}_{si}\right)\right] dvdt \tag{6}$$

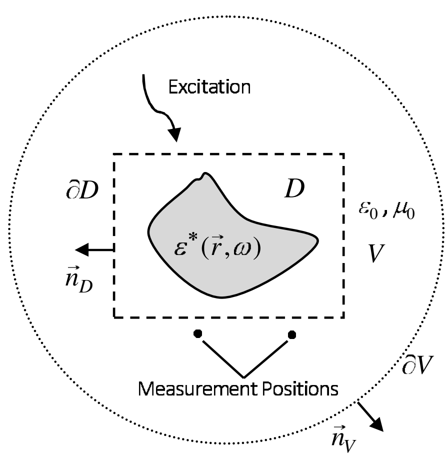


Figure 1: Geometry of the inverse scattering problem.

and PC is the term related to the polarization equality constraint and is given by

$$PC = \sum_{i=1}^{I} \int_0^T \int_V \vec{q}_i \cdot \left(\vec{J}_i + \tau \partial_t \vec{J}_i - \varepsilon_0 \Delta\varepsilon \partial_t \vec{E}_i \right) dvdt \quad \text{(Debye case)}$$
$$PC = \sum_{i=1}^{I} \int_0^T \int_V \vec{q}_i \cdot \left(\omega_0^2 \vec{J}_i + 2\zeta \partial_t \vec{J}_i + \partial_t^2 \vec{J}_i - \varepsilon_0 \Delta\varepsilon \omega_0^2 \partial_t \vec{E}_i \right) dvdt \quad \text{(Lorentz case)} \tag{7}$$

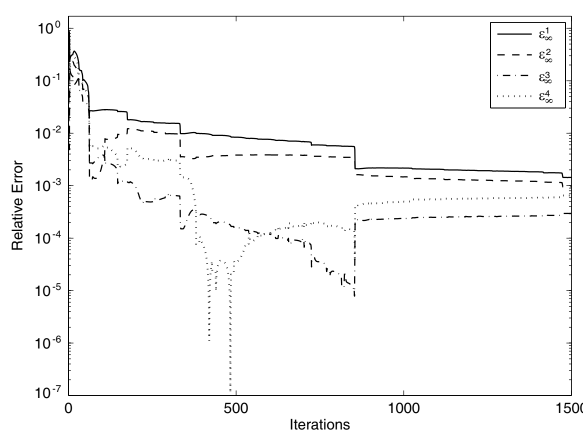
In (6) and (7), $\vec{h}_i$, $\vec{e}_i$, and $\vec{q}_i$ are the vectorial Lagrange muliplиers.

2.3. Vectorial Lagrange Multipliers — Fréchet Derivatives

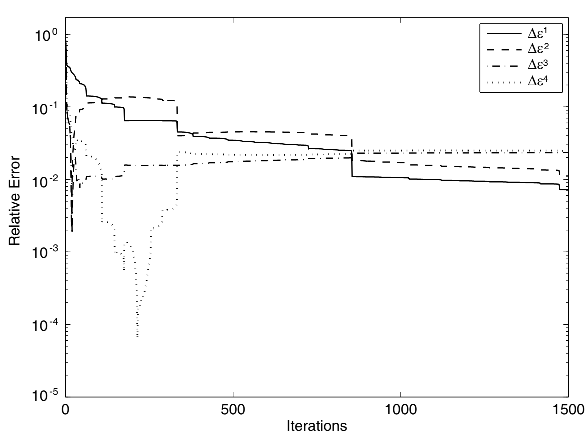
Using the calculus of variations, the stationarity condition of the cost functional, $\delta F = 0$, results in the differential equations (adjoint problem) satisfied by the vectorial Lagrange multipliers, i.e.,

$$\nabla \times \vec{e}_i - \mu \partial_t \vec{h}_i = 0, \quad \nabla \times \vec{h}_i + \varepsilon_0 \varepsilon_\infty \partial_t \vec{e}_i - \vec{j}_i + \sum_{k=1}^{K} (\vec{E}_{ik} - \vec{E}_{ik}^m) = 0, \tag{8}$$

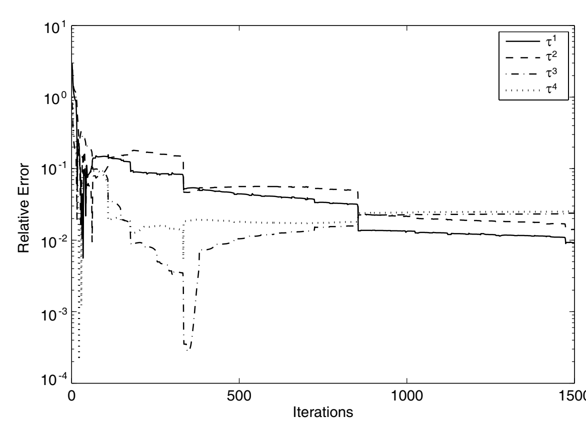
$$\vec{j}_i - \tau \partial_t \vec{j}_i + \varepsilon_0 \Delta\varepsilon \partial_t \vec{e}_i = 0, \quad \vec{j}_i = -\varepsilon_0 \Delta\varepsilon \partial_t \vec{q}_i \quad \text{(Debye case)}$$
$$\omega_0^2 \vec{j}_i - 2\zeta \partial_t \vec{j}_i + \partial_t^2 \vec{j}_i + \varepsilon_0 \Delta\varepsilon \omega_0^2 \partial_t \vec{e}_i = 0, \quad \vec{j}_i = -\varepsilon_0 \Delta\varepsilon \omega_0^2 \partial_t \vec{q}_i \quad \text{(Lorentz case)} \tag{9}$$



(a)



(b)



(c)

Figure 2: Relative reconstruction error vs. the number of iterations, for the case of 2D Debye scatterer divided in four subdomains. (a) ε_∞, (b) $\Delta\varepsilon$, and (c) τ.

Furthermore, the Fréchet derivatives of the cost functional with respect to the scatterer properties are derived analytically and are given by

$$\frac{\delta F}{\delta \varepsilon_\infty} = -\varepsilon_0 \sum_{i=1}^{I} \int_0^T \left(\vec{e}_i \cdot \partial_t \vec{E}_i\right) dt,\ \frac{\delta F}{\delta \Delta\varepsilon} = -\frac{1}{\Delta\varepsilon} \sum_{i=1}^{I} \int_0^T \left(\vec{E}_i \cdot \vec{j}_i\right) dt,\ \frac{\delta F}{\delta \tau}$$
$$= \frac{1}{\varepsilon_0 \Delta\varepsilon} \sum_{i=1}^{I} \int_0^T \left(\vec{J}_i \cdot \vec{j}_i\right) dt, \tag{10}$$
$$\frac{\delta F}{\delta \omega_0} = 2 \sum_{i=1}^{I} \int_0^T \left[\frac{1}{\omega_0} \vec{J}_i \cdot \vec{e}_i - \frac{\left(2\zeta \vec{j}_i - \partial_t \vec{j}_i\right) \cdot \vec{J}_i}{\varepsilon_0 \omega_0^3 \Delta\varepsilon}\right] dt,\ \frac{\delta F}{\delta \zeta} = \frac{2}{\varepsilon_0 \omega_0^2 \Delta\varepsilon} \sum_{i=1}^{I} \int_0^T \left(\vec{J}_i \cdot \vec{j}_i\right) dt.$$

The Fréchet derivatives in (10) can be utilized by any gradient-based optimization algorithm to update the dispersive scatterer properties, iteratively. In this work, the Polak-Ribière optimization algorithm is adopted, while the computation of the estimated fields and the vectorial Lagrange multipliers is based on the FDTD method.

3. NUMERICAL RESULTS

The presented method has been applied to the reconstruction of a two-dimensional Debye scatterer. The incident waves are generated by eight transmitters ($I = 8$) placed at uniformly distributed locations around the scatterer. For each incidence, the electric field is measured at sixteen positions ($K = 16$), uniformly placed around the scatterer. The scatterer is considered to be square,

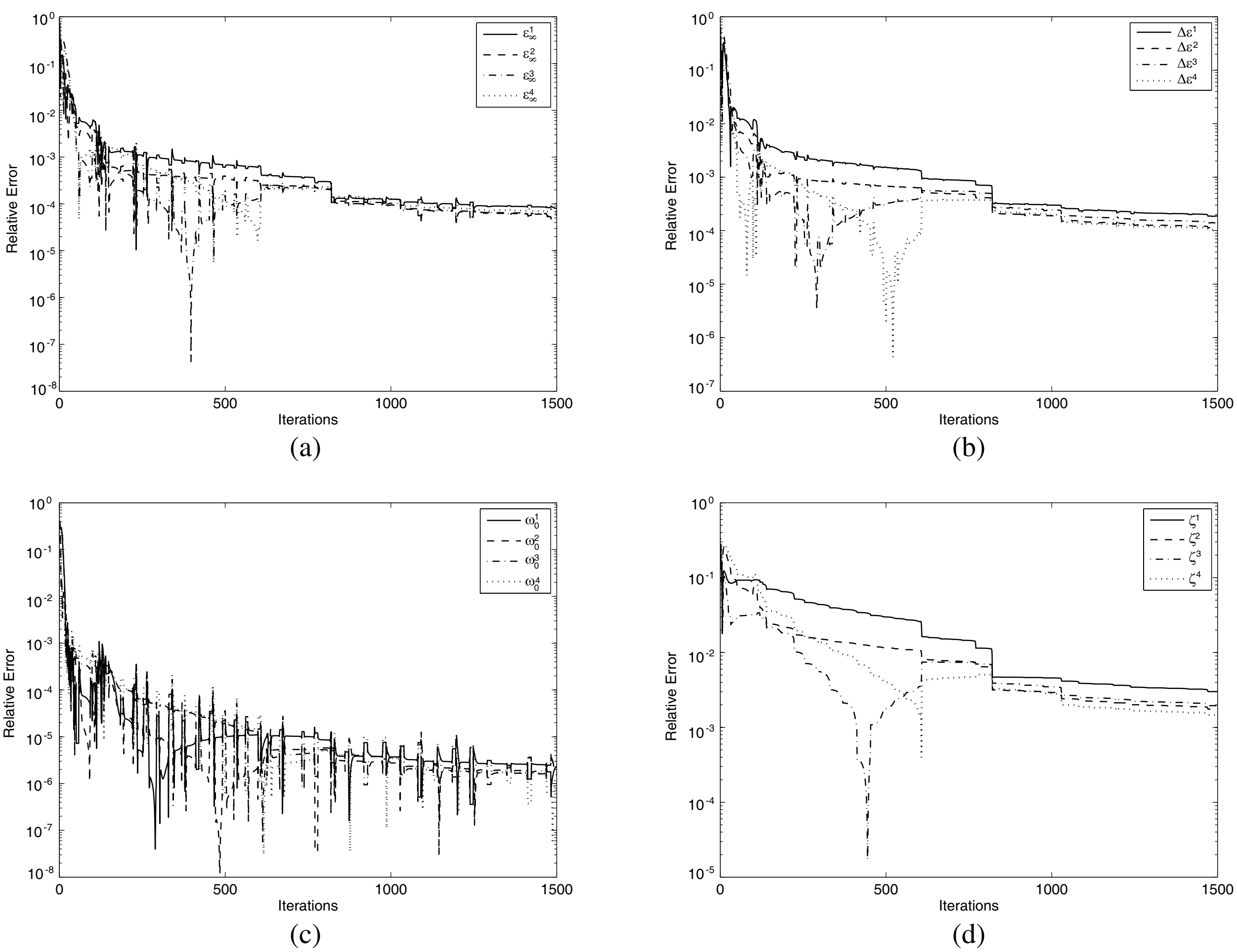


Figure 3: Relative reconstruction error vs. the number of iterations, for the case of 2D Lorentz scatterer divided in four subdomains. (a) ε_∞, (b) $\Delta\varepsilon$, (c) ω_0, and (d) ζ.

(0.75m $\times$ 0.75m), and is divided in four square subdomains characterized by different parameters. Specifically, the original properties of the scatterer subdomains are $[\varepsilon_\infty^1, \varepsilon_\infty^2, \varepsilon_\infty^3, \varepsilon_\infty^4] = [2, 3, 4, 2]$, $[\Delta\varepsilon^1, \Delta\varepsilon^2, \Delta\varepsilon^3, \Delta\varepsilon^4] = [3, 2, 2, 2]$, and $[\tau^1, \tau^2, \tau^3, \tau^4] = [0.5, 0.5, 1, 1]\,\text{ns}$. The excitation current density is a modulated Gaussian pulse given by

$$J_s(t) = e^{-a^2(t-4/a)^2} \sin(2\pi f_c t) u(t) \tag{11}$$

where $f_c = 5 \times 10^8\,\text{Hz}$ and $a = 5\pi \times 10^8\,\text{sec}^{-1}$.

The scatterer reconstruction is based on simulated measurements obtained by the method of the FDTD. The values of the initial estimates of the scatterer parameters are $\varepsilon_\infty^{1,2,3,4} = 1.1$, $\Delta\varepsilon^{1,2,3,4} = 0.5$, and $\tau^{1,2,3,4} = 2\,\text{ns}$. In addition, the widths of the four subdomains are *a priori* known. After 1500 iterations the cost function is lower than 2×10^{-4}, while the relative reconstruction errors are of order or lower than 10^{-2}, for all three Debye parameters; a fact that depicts the efficiency of the proposed technique. Figure 2 presents the relative reconstruction errors for each parameter, ε_∞, $\Delta\varepsilon$, τ, versus the number of iterations of the algorithm.

Finally, the proposed microwave imaging method is applied to the reconstruction of a two-dimensional Lorentz dispersive scatterer. The geometry of the inverse problem, the excitation and the measurement positions are the same as in the previous Debye case. The original properties of the scatterer subdomains are $[\varepsilon_\infty^1, \varepsilon_\infty^2, \varepsilon_\infty^3, \varepsilon_\infty^4] = [2, 2.5, 2.5, 2]$, $[\Delta\varepsilon^1, \Delta\varepsilon^2, \Delta\varepsilon^3, \Delta\varepsilon^4] = [2, 3, 2.5, 3]$, $[\omega_0^1, \omega_0^2, \omega_0^3, \omega_0^4] = [\pi \times 10^9, \pi \times 10^9, \pi \times 10^9, \pi \times 10^9]\,\text{rad/sec}$, and $[\zeta^1, \zeta^2, \zeta^3, \zeta^4] = [0.08\pi \times 10^8, 0.08\pi \times 10^8, 0.09\pi \times 10^8, 0.09\pi \times 10^8]\,\text{sec}^{-1}$. The excitation current density is the same as in the previous example.

The values of the initial estimates of the scatterer parameters are $\varepsilon_\infty^{1,2,3,4} = 3.5$, $\Delta\varepsilon^{1,2,3,4} = 1.1$, $\omega_0^{1,2,3,4} = 6\pi \times 10^8\,\text{rad/sec}$, $\zeta^{1,2,3,4} = 0.06\pi \times 10^8\,\text{sec}^{-1}$ and the unknowns are now 16. As already noted, the size of the four subdomains are *a priori* known. After 1500 iterations the cost function is lower than 3×10^{-5}, while the relative reconstruction errors are of order or lower than 10^{-3}, for all four Lorentz parameters. Figure 3 presents the relative reconstruction errors for each parameter, ε_∞, $\Delta\varepsilon$, ω_0, and ζ versus the number of iterations of the algorithm.

4. CONCLUSION

A time-domain microwave imaging technique, which is based on the use of vectorial Lagrange multipliers, has been proposed for the reconstruction of dispersive scatterers. The Fréchet derivatives of the cost functional with respect to the scatterer properties are derived analytically. These derivatives can be utilized by any gradient-based inverse scattering technique along with any time-domain computational method employed for the solution of the electromagnetic problem. The cases of two-dimensional Debye as well as Lorentz dispersive scatterers have been investigated. Numerical results have shown that it is possible to reconstruct the spatial distributions of all the electromagnetic properties of the scatterers, accurately and simultaneously.

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REFERENCES

1. Pastorino, M., *Microwave Imaging*, Wiley, New Jersey, 2010.
2. Colton, D. and R. Kress *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, New York, 1992.
3. Pastorino, M., S. Caorsi, and A. Massa, "A global optimization technique for microwave non-destructive evaluation," *IEEE Trans. Instrum. Meas.*, Vol. 51, No. 4, 666–673, 2002.
4. Meaney, P. M., K. D. Paulsen, and T. P. Ryan, "Two-dimensional hybrid element image reconstruction for TM illumination," *IEEE Trans. Antennas Propag.*, Vol. 43, No. 3, 239–247, 1995.
5. Joachimowicz, N., C. Pichot, and J.-P. Hugonin, "Inverse scattering: An iterative numerical method for electromagnetic imaging," *IEEE Trans. Antennas Propag.*, Vol. 39, No. 12, 1742–1752, 1991.

6. Gustafsson, M. and S. He, "An optimization approach to two-dimensional time domain electromagnetic inverse problems," *Radio Sci.*, Vol. 35, No. 2, 525–536, 2000.
7. Takenaka, T., H. Zhou, and T. Tanaka, "Inverse scattering for a three-dimensional object in the time domain," *JOSA A*, Vol. 20, No. 10, 1867–1874, 2003.
8. Rekanos, I. T., "Time-domain inverse scattering using Lagrange multipliers: An iterative FDTD-based optimization technique," *Journal of Electromagnetic Waves and Applications*, Vol. 17, No. 2, 271–289, 2003.
9. Papadopoulos, T. G. and I. T. Rekanos, "Estimation of the parameters of Lorentz dispersive media using a time-domain inverse scattering technique," *IEEE Trans. Magn.*, Vol. 48, No. 2, 219–222, 2012.
10. Papadopoulos, T. G. and I. T. Rekanos, "Time-domain microwave imaging of inhomogeneous Debye dispersive scatterers," *IEEE Trans. Antennas Propag.*, Vol. 60, No. 2, 1197–1202, 2012.