# Dispersion Characteristic Analysis of Open Cylindrical Waveguide and Its Metalic Closed Model

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**Abstract**— In the study, complex modes dispersion characteristics of cylindrical plasma column open waveguide are obtained from exact characteristic equations system and its metallic closed model. In order to model the open waveguide with cylindrical plasma column loaded closed waveguide, firstly mathematically proper guided and complex modes are obtained. Because mathematically proper modes do not violate the boundary conditions and can be modeled with a closed waveguide structure which uses a complete set of orthogonal functions. Numerical values obtained from characteristic equations system and model structure for cylindrical isotropic plasma column open waveguide are presented in the same figure. Additionally, it is shown that the numerical results obtained from the model approach to the exact solution while radius of metallic waveguide frame gets larger.

## 1. INTRODUCTION

Complex waves have been investigated in numerous studies for closed and open waveguides. Complex wave modes for isotropic dielectric rod loaded closed waveguide and parallel isotropic dielectric and anisotropic ferrite sheet were presented by Clarricoats [1] and Marques [2], respectively. For open waveguides, leaky waves are the most popular complex waves. The reason is that they play very important roles at antennas, microwave circuits and a large variety of physical phenomenon [3]. The leaky wave modes are physically improper. They are not member of a complete set of orthogonal base functions. This solution for the wave equation gives a field presentation with complex propagation constant, but that the field becomes infinite at the infinite transverse spatial limit [4]. Improper nature of leaky waves originates from increase of the field while leaky wave diverges transversely from guide. If the wave exists in free space, amplitude of the wave will diverge transversely at infinity and therefore, boundary conditions at infinity will be violated. Hence these waves are characterized mathematically improper or nonspectral. It can be shown easily that the wave never arrives transverse infinity for a distance from source to infinite and it is described only within near space sector of guiding structure. It is stated in [3] that leaky wave is certainly physical and measurable in valid region and it is really a type of near field wave. Nevertheless it affects strongly nature of far field.

In our study, complex dispersion characteristics of an open waveguide is obtained by using its metallic closed model. Equations, which belong to metallic closed model, are obtained by using Method of Moment (MoM) based on generalized telegraphist's equations. Even though leaky wave modes have very large implementation field and are an important physical phenomenon, their mathematically improper nature prevents modeling these modes of open waveguide with the closed waveguide structure. Because boundary conditions for metallic closed waveguides, whose tangential component of electric field and normal component of magnetic field equal to zero at metallic wall, automatically are violated. Besides transmission line equations used in modeling consist of a complete set of orthogonal base functions [5]. The conditions which leaky waves violate the boundary conditions at infinite and they are not a member of complete set of orthogonal base functions, prevent to work the model.

In our study, firstly, mathematically proper modes are obtained, because mathematically proper modes enable to be modeled with a closed waveguide structure which uses a complete set of orthogonal functions. Additionally they do not violate the boundary conditions when the open waveguide is closed with a metallic wall. Because the fields decrease while diverging from the guiding structure. In the study,cylindrical isotropic plasma column open waveguide is taken as an example structure. Complex modes for cylindrical dielectric rod open waveguide were presented firstly by Arnbak. Later, complex modes for cylindrical dielectric rod open waveguide and cylindrical plasma column open waveguide were investigated by using Davidenko's method [6–8]. But complex modes of open waveguide structures in the studies are mathematically improper leaky modes. In the literature, mathematically proper complex modes of cylindrical isotropic plasma column open waveguide could not be found. Guided waves of the structure were presented in [9] and they are obtained again from characteristic equation system of the structure in our study. Complex modes which correspond to mathematically proper modes of the structure are obtained from the same characteristic equation system by using "fslove" function in Matlab.

In our study, propriety or impropriety of modes of cylindrical plasma column open waveguide are investigated analytically. It is shown that modes obtained from numerical computations correspond to proper modes. Analytic solutions of cylindrical isotropic plasma column loaded closed waveguide used as model structure in the study are obtained by using generalized telegraphist's equations (or transmission line equivalent equations) presented in [5]. This approximation to model open waveguide with closed metallic guide was firstly used to model guided waves of rectangular dielectric open waveguide by Ogusu [10]. In our study, numerical values obtained from characteristic equations system and model structure for cylindrical isotropic plasma column open waveguide are presented in the same figure. Additionally, it is shown that the numerical results obtained from the model approach to the exact solution while radius of metallic waveguide frame gets larger.

## 2. CYLINDRICAL PLASMA COLUMN OPEN WAVEGUIDE

Plasma called fourth state of matter is an ionized state of matter, which exists in universe and can be obtained artificially in the earth. Plasma medium is an anisotropic medium. But it is typically tackled under special conditions which transform anisotropic medium into isotropic medium. Thus supplies simplification in solutions. One of these simplifications is that the radian electron cyclotron frequency ($\omega_c$) is taken zero, which is called plasma column with zero magnetic field in the literature [9]. In our study, isotropic plasma medium with frequency depended permittivity, $\tilde{\varepsilon} = \varepsilon_0(1 - \frac{\omega_p^2}{\omega^2}) = \varepsilon_0\varepsilon_p$ is investigated by using the simplification. Where, $\omega$ is the angular frequency, $\omega_p$ is the radian electron plasma frequency and $\varepsilon$ is free space permittivity.

In [9], dispersion curves of zero cyclotron frequency condition were presented for parametric values, azimuthal variation $m = 0$, 1, 2 and normalized plasma frequency ($\omega_p$ a/c) $\omega_N = 0.25$, 0.5, 0.75. Where, a indicates plasma column radius and $c$ indicates speed of light in vacuum

Analytic solutions for cylindrical isotropic plasma column open waveguide surrounded with free space are obtained from Maxwell equations by using continuity conditions at interface between plasma-air. Variations of the fields is described as below.

$$F(r, \varphi, z) = F(r)\, e^{j(\omega t - \gamma z - m\varphi)} \tag{1}$$

Here, $\gamma$ shows complex propagation constant and is described as $\gamma = \beta - j\alpha$. Here, $\alpha$ is attenuation constant and $\beta$ is phase constant.

For the field variation described in Equation (1), characteristic equations system is obtained as in Equation (2) for lossless and source less cylindrical plasma column open waveguide with $\varepsilon_0\varepsilon_p$ permittivity and $\mu$ permeability. Here, $\mu_0$ is free space permeability. In Equation (2), the points at which the determinant of the coefficient matrix vanishes, give the solution

$$\begin{bmatrix} -\frac{m}{a}\frac{\gamma}{k_1^2}J_m(k_1a) & j\frac{1}{\omega\varepsilon_0}\frac{k_0^2}{k_1}J'_m(k_1a) & -\frac{m}{a}\frac{\gamma}{k_2^2}K_m(k_2a) & j\frac{1}{\omega\varepsilon_0}\frac{k_0^2}{k_2}K'_m(k_2a) \\ 0 & J_m(k_1a) & 0 & -K_m(k_2a) \\ -j\frac{1}{\omega\mu_0}\frac{k_0^2\varepsilon_p}{k_1}J'_m(k_1a) & -\frac{m}{a}\frac{\gamma}{k_1^2}J_m(k_1a) & -j\frac{1}{\omega\mu_0}\frac{k_0^2}{k_2}K'_m(k_2a) & -\frac{m}{a}\frac{\gamma}{k_2^2}K_m(k_2a) \\ J_m(k_1a) & 0 & -K_m(k_2a) & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0 \tag{2}$$

where, $J_m$ is Bessel function of first kind with $m$th degree, $K_m$ is modified Bessel function of second kind with $m$th degree. Besides, $k_1^2$, $k_2^2$ and $k_0^2$ are descirbed as $k_1^2 = k_0^2\varepsilon_p - \gamma^2$, $k_2^2 = \gamma^2 - k_0^2$ and $k_0^2 = \omega^2\mu_0\varepsilon_0$ respectively.

Complex dispersion curves generally include dispersion characteristics of forward and backward guided waves, evanescent and complex waves. In numerical computation, a noncomplex root which satisfies Equation (2) for fixed working frequency, can be easily found with simple methods as bisection method. Nevertheless obtaining complex roots gets harder while the characteristic equation gets complicated. In the study complex modes of cylindrical plasma column open waveguide are obtained from the characteristic equations system by using "fsolve" function in Matlab. "fslove" function in Matlab requires an approximate initial value to compute noncomplex or complex root. In order to determine initial value, the complex modes reported in the literature are taken as reference. The complex waves are presented for different structures as cylindrical dielectric rod loaded closed waveguide [11], parallel dielectric and anisotropic ferrite sheet waveguide [2]. In these studies, common feature of complex modes is that complex modes arise at points where slope of the dispersion curves changes sign. In respect ofthis feature, complex modes of our structure are computed about the frequency point where slope of dispersion curve changes sign by using Equation (2) and "fsolve" function in Matlab.

### 2.1. Analysis of Proper and Improper Complex Modes of Plasma Column Open Waveguide

Field components of open waveguides have to supply boundary conditions at infinity. Boundary conditions at infinity are given below.

$$\lim_{R\to\infty} RE = \lim_{R\to\infty} RH = 0$$

Here, $R$ is determined as $R = \sqrt{r^2 + z^2}$, $r$ and $z$ are cylindrical coordinate variables.

If amplitude of wave increases while $r \to \infty$, the boundary conditions at infinity is violated and the wave is called mathematically improper. Nevertheless these waves also called leaky waves can be measurable and used in numerous electromagnetic applications. But, since leaky waves are mathematically improper,they can only exist in a restricted region of a wedge shape within which the field stays finite [6].

The characteristic equations system given in Equation (2) of cylindrical plasma column open waveguide is obtained from Maxwell equations and continuity conditions between plasma and air. All field expressions for region outside of plasma column consist of modified Bessel function of second kind or its derivatives. Since derivatives of Bessel functions can be expressed in the terms of themselves if we take limit of modified Bessel function of second kind while $r \to \infty$, we can obtain analytic results about proper or improper of fields. While $r \to \infty$, asymptotic expression of modified Bessel function of second kind is given below.

$$(kr) \simeq \frac{e^{-kr}}{\sqrt{2\pi kr}} \tag{3}$$

where, $k$ is Bessel function parameter and called the wave number. Let's investigate Equation (3) for three cases of complex wave number, $\mathrm{Re}(k) = k_R$ ve $\mathrm{Im}(k) = k_I (k = k_R + jk_I)$. If the equation is investigated for pure real, pure imaginary and complex $k$ while $r \to \infty$, the results below are obtained.

$$k = k_R \text{ (pure real) } \rightarrow \begin{cases} k_R > 0 & \text{proper mode} \\ k_R < 0 & \text{improper mod.} \end{cases}$$

$$k = jk_I \text{ (pure imaginary) } \rightarrow \text{ proper mode (independent of value of } k_I)$$

$$k = k_R + jk_I \text{ (complex)} \rightarrow \text{ (independent of value of} k_I) \quad \begin{cases} k_R > 0 & \text{proper mode} \\ k_R < 0 & \text{improper mode .} \end{cases}$$

Nevertheless the wave number outside of plasma column which is parameter of modified Bessel function of second kind depends on angular frequency and the propagation constant $k_2^2 = \gamma^2 - k_0^2$ constant. For real working frequency, the free space wave number $k^2$ is always real. Accordingly, values of $k_2$ depending on the propagation constant and propriety and impropriety of modes is achieved as below.

$$\gamma \text{ pure real } \begin{cases} \gamma^2 > k_0^2 \Rightarrow k_2 = \pm|\lambda| \begin{cases} +|\lambda| & \text{positive real (proper mode)} \\ -|\lambda| & \text{negative real (improper mode)} \end{cases} \\ \gamma^2 < k_0^2 \Rightarrow k_2 \text{ pure imaginary (proper mode).} \end{cases}$$

$$\gamma \text{ pure imaginary (} \gamma^2 \text{ is negative real) ,} \quad \gamma^2 - k_0^2 < 0 \text{ and } k_2 \text{ is pure imaginary (proper mode)}$$

$$\gamma \text{ complex (} k_2 \text{ complex)} \quad \begin{cases} \mathrm{Re}\{k_2\} > 0 \Rightarrow & \text{proper mode} \\ \mathrm{Re}\{k_2\} < 0 \Rightarrow & \text{improper mode .} \end{cases}$$

### 2.2. Exact Dispersion Characteristic of the Structure

In this section, exact dispersion curves of cylindrical plasma column open waveguide are obtained from characteristic equations system given in Equation (2). Additionally, it is shown numerically that the modes obtained from computations correspond to proper modes.

Parametric values of cylindrical plasma column open waveguide used in numerical computations are that plasma column radius ($a$) is 10 mm, azimuthal variation ($m$) is 1 and normalized plasma frequency ($\omega_p$) is 0.25. In numerical computations, firstly, propagation constants for a fixed frequency are obtained. Later the wave number outside of plasma column ($k_2$) are computed by using these propagation constants. Dispersion curves and wave number outside of plasma column which determine propriety or impropriety of wave are given together in Figure 1.

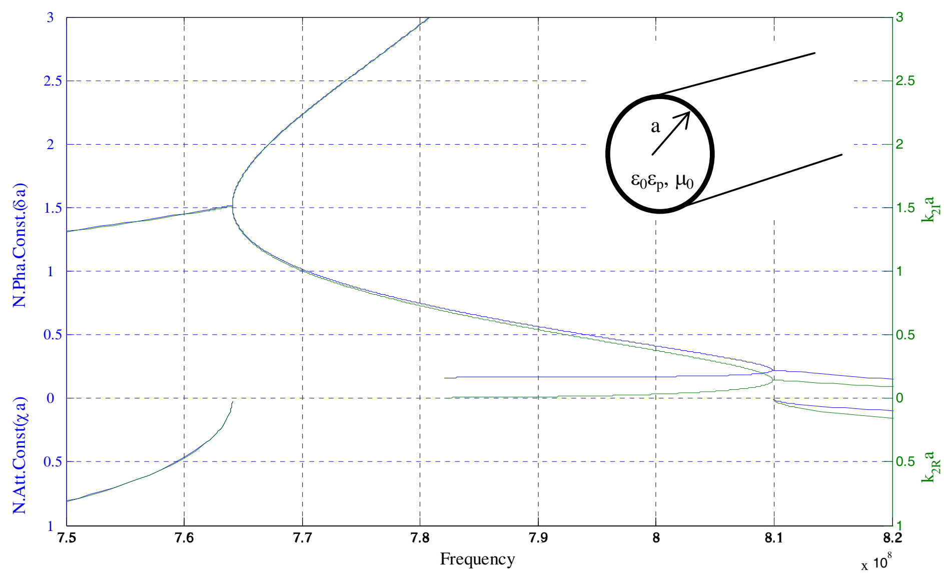


Figure 1: Left $y$ axis: Complex dispersion curves for cylindrical isotropic plasma column open waveguide. Right $y$ axis: Normalized wave number outside of plasma column. Inset: geometry of the problem.

In the figure, above from zero point at left $y$ axis shows normalized phase constant ($\beta a$) and below shows normalized attenuation constant ($\alpha a$). Similarly,above from zero point at right $y$ axis shows imaginary part of normalized wave number outside of plasma column ($k_{2I}a$) and below shows real part of normalized wave number outside of plasma column ($k_{2R}a$). As shown in the figure, below from zero point are positive at both sides indicating proper nature of the complex mode observed.

As shown from blue curves in the figure, the complex modes arise at the frequency points where slope of dispersion curve changes sign. For the structure, the dispersion curve bifurcates at the frequency point 764 MHz. The complex modes exist on the left side of this point, forward and backward guided waves exist on the right side of this point. Similarly, the dispersion curve bifurcates at the frequency point 810 MHz. Forward and backward wave modes exist on the left side of this point and complex modes exist on the right side of this point. As shown from green curves in the figure, real parts of normalized wave number outside of plasma column are positive. As a result, for cylindrical plasma column open waveguide, guided modes and complex modes obtained from numerical computations and presented in Figure 1 correspond to proper modes.

## 3. MOM MODEL OF THE STRUCTURE

One of the strongest methods for analyzing closed waveguide structures which do not have exact solution was given by Schelkunoff [5]. In the study, Schelkunoff transformed Maxwell equations that consist of partial differential equations into an ordinary differential equations system by modeling the waveguide as an infinite system of transmission line equations. The ordinary differential equations system transforms into algebraic equations system, because derivatives with respect to propagation direction variable correspond to product with a fixed value ($-j\gamma$) for the field variation in Equation (1). So, approximate solution of the structure which does not have analytic solution can be obtained. This method is also called Method of Moment (MoM). In this section, modes of cylindrical plasma column open waveguide is obtained from model of cylindrical plasma column loaded closed waveguide. This approximation was applied to obtain guided wave characteristics of rectangular dielectric open waveguide by Ogusu [10]. In order to apply the method for our structure, cylindrical plasma column surrounded with air is closed within a metallic wall.

Investigated modes have to be mathematically proper for validation of the method. Modeling for MoM is performed by using orthogonal base functions which belong to metallic empty closed guide. The boundary conditions belonging to the closed structure are that tangential component of electric field and normal component of magnetic field on the metallic boundary are zero. Improper modes violate the boundary conditions of closed waveguide, because amplitude of their field component increase while diverging transversely from plasma column. Additionally, they are not a member of a complete set of orthogonal base functions [4]. Because amplitude of field component of proper modes decrease while diverging transversely from plasma column, if the metallic wall is placed far enough from plasma column, it is possible to model proper modes of plasma column open waveguide with closed structure. Forward and backward guided wave modes and complex wave mode, which constitute the dispersion curves of cylindrical plasma column open waveguide given in Figure 1, correspond to mathematically proper modes. Hence it is possible to model them by enclosing the plasma column in a metallic shield, since then the boundary condition at infinity can be satisfied.

## 4. NUMERICAL RESULTS

In this section, all modes (guided and complex) presented in Section 2 for cylindrical plasma column open waveguide are obtained by using MoM solutions belonging to cylindrical plasma column loaded closed waveguide. Results obtained from MoM for different radius values of metallic wall and from exact solution for cylindrical plasma column open waveguide are presented together in the same figure. Computations for MoM are performed by using 150 TM and 150 TE modes of empty metallic closed waveguide.

In Figure 2, dispersion curves of cylindrical plasma column open waveguide obtained from exact solution with the same parametric values used in Figure 1 and dispersion curves obtained from MoM for $b = 20\,\mathrm{mm}$ (radii ratio 0.5), $b = 33\,\mathrm{mm}$ (radii ratio 0.333) and $b = 100\,\mathrm{mm}$ (radii ratio 0.1) are presented altogether.

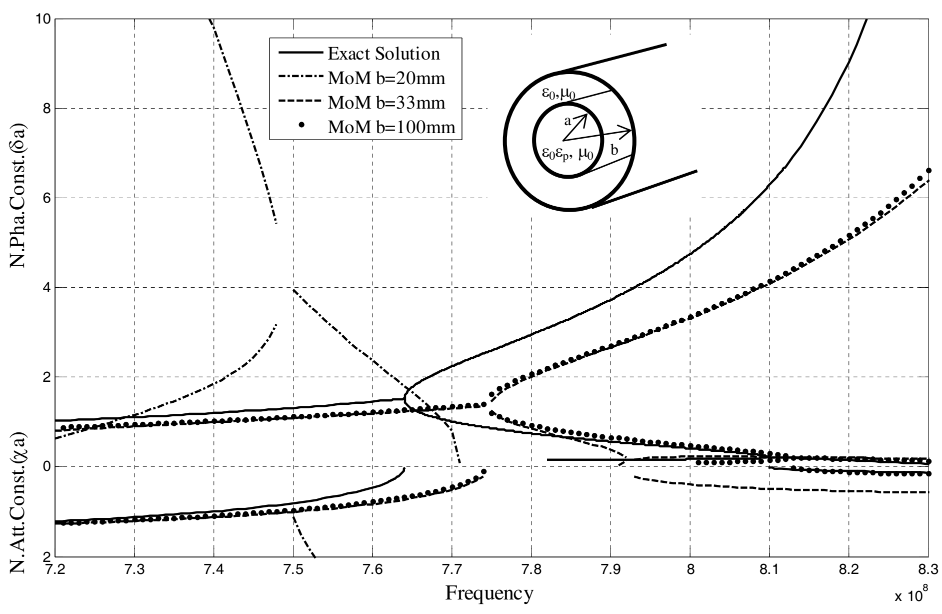


Figure 2: Dispersion curves obtained from exact solution and MoM model open plasma column.

As presented in the figure, the results obtained from MoM for $b = 20\,\mathrm{mm}$ are rather far from results obtained from exact solution. Nevertheless results obtained by MoM is getting close to the exact solution while the ratio of radii is decreasing or radius of metallic wall is getting larger. For exact dispersion curve which bifurcates in the vicinity of 774 MHz, dispersion curves obtained from MoM for $b = 33\,\mathrm{mm}$ and $b = 100\,\mathrm{mm}$ are nearly at the same distance to the exact dispersion curve. But dispersion curve obtained from MoMfor $b = 100\,\mathrm{mm}$ are much more compatible with the exact solution for bifurcation point in the vicinity of 810 MHz.

Consequently, dispersion curves obtained from MoM for different values of metallic wall radius gets more compatible with exact solution while the radius is increased, as shown in Figure 2.

Nevertheless it is necessary to use more TM and TE base functions at computations of MoM when the metallic wall radius is much increased. It is obvious that computations performed with more base functions cause to increase computational load and time. As a result of computations performed by using 150 TE and 150 TM base functions, the most suitable ratio of radii has been determined as 0.1.

## 5. CONCLUSION

In this study, it is aimed to obtain dispersion curves of open waveguide structure for a wide spectrum including mathematically proper complex waves from exact solution and MoM model of the structure. Cylindrical isotropic plasma column open waveguide whose forward and backward guided wave modes were presented in the literature is tackled as example structure. Guided modes and complex modes are obtained from exact characteristic equations system of the structure. The structure is transformed to cylindrical plasma column loaded closed wave guide for modeling by closing the open structure with metallic wall. MoM equations based generalized telegraphest's equations are obtained for the closed structure. Numerical results for different radius values of metallic wall are performed by using these equations. It is shown by presenting exact results and MOM results in the same figure that results obtained from MoM is getting more compatible with the exact solution while metallic wall radius is getting larger.

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