On the Coneigenvalue Decomposition of Sinclair Matrices

T. Dallmann and D. Heberling
Institute of High Frequency Technology, RWTH Aachen University, Germany

Abstract— The polarimetric response of radar targets can be decomposed into a set of so-called Huynen-Euler parameters which allow to analyze the physical scattering mechanism occuring at the target. Required for this technique is a coneigenvalue decomposition, whose characteristics are not yet fully unterstood. Due to this there are conflicting opinions regarding the question if one of the Huynen-Euler parameters, the skip angle, can be extracted from the coneigenvalues. This paper resolves this conflict by discussing the mathematics of coneigenvalues and by giving a physical interpretation to the coneigenvalue decomposition.

1. INTRODUCTION

The Sinclair matrix is a 2×2 matrix which can be used to describe the polarimetric backscattering of radar targets. By decomposing this matrix into the so-called Huynen-Euler parameters various scattering mechanisms occuring at the target can be identified. The extraction of the Huynen-Euler parameters requires a coneigenvalue decomposition of the Sinclair matrix $\underline{\mathbf{S}}$ which can be described with [1]

$$\underline{\mathbf{S}} = \underline{\mathbf{U}}^* \; \underline{\mathbf{\Sigma}} \; \underline{\mathbf{U}}^H \tag{1}$$

where the diagonal matrix $\underline{\mathbf{\Sigma}}$ can be defined to

$$\underline{\mathbf{\Sigma}} = m\, e^{j\xi} \begin{bmatrix} e^{+j2\nu} & 0 \\ 0 & e^{-j2\nu} \tan^2\gamma \end{bmatrix} \tag{2}$$

and the unitary matrix $\underline{\mathbf{U}}$ consists of

$$\underline{\mathbf{U}} = \underline{\mathbf{U}}_\psi \, \underline{\mathbf{U}}_{\tau_m} \, \underline{\mathbf{U}}_\alpha = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \cos\tau_m & j\sin\tau_m \\ j\sin\tau_m & \cos\tau_m \end{bmatrix} \begin{bmatrix} e^{+j\alpha} & 0 \\ 0 & e^{-j\alpha} \end{bmatrix}. \tag{3}$$

The Huynen-Euler parameters are the maximum radar cross-section (RCS) m, the orientation angle ψ, the helicity angle τ_m, the polarizability γ, the absolute phase ξ and — at least if following the definition found in literature [1, 2] — the skip angle ν. These parameters allow to distinguish between scattering events like single reflections, double reflections and dipole scattering and provide even more information like the orientation of the target. Since these events not only influence the polarization of the scattered wave but also indicate at which angles and frequencies the RCS of a target increases, this set of parameters is of special interest for radar cross-section analysis. Therefore this decomposition is used in some recent publications about the measurement and analysis of RCS signatures and radar images [3, 4].

Unfortunately one of the Huynen-Euler parameters causes problems: There are conflicting opinions regarding the question if the skip angle ν which allows the distinction between single and double reflections can be determined from the Sinclair matrix. For the entries of the diagonal matrix $\underline{\mathbf{\Sigma}}$ in (1) it can be shown that the phase of the diagonal entries is indetermined. On the first sight this prevents that the skip angle can be determined properly and led Lüneburg to following conclusion:

> "The phase indeterminacy of the coneigenvalue is an essential feature of the antilinear time-reversal operation in backscattering. Its interpretation and significance for target characterization and classification purposes (Huynen's skip angle) is at present not fully understood." [5]

In contrast to this a technique was developed by Baird which allows to extract Huynen's skip angle from the Sinclair matrix [4]. To the knowledge of the authors this discrepancy is still an unresolved problem.

To resolve the aforementioned contradiction the connection between coneigenvalues and coneigenvectors will be explained in this paper. For comparison the diagonalization of Jones matrices which is based on the eigenvalue decomposition is explained first. Afterwards, a physical interpretation of the coneigenvalues of the Sinclair matrix will be given. Finally the role of the skip angle will be emphasized. The explanations and approaches discussed in this paper therefore contribute to the effort of placing radar polarimetry on a solid mathematical foundation.

2. DIAGONALIZATION OF JONES MATRICES

The Jones matrix $\underline{\mathbf{J}}$ is similar to a diagonal matrix $\underline{\mathbf{\Lambda}}$ via [5]:

$$\underline{\mathbf{U}}^H\,\underline{\mathbf{J}}\,\underline{\mathbf{U}} = \underline{\mathbf{\Lambda}} \tag{4}$$

Here $\underline{\mathbf{U}}$ is a unitary matrix. Rewriting (4) by using $\underline{\mathbf{U}} = [\underline{\mathbf{u}}_1 \;\; \underline{\mathbf{u}}_2]$ it can be seen that $\underline{\mathbf{\Lambda}}$ contains the eigenvalues of $\underline{\mathbf{J}}$:

$$\begin{array}{rlcll} & \underline{\mathbf{J}}\,\underline{\mathbf{U}} & = & \underline{\mathbf{U}}\,\underline{\mathbf{\Lambda}} & \\ \Rightarrow & \underline{\mathbf{J}}[\underline{\mathbf{u}}_1 \;\; \underline{\mathbf{u}}_2] & = & [\underline{\mathbf{u}}_1 \;\; \underline{\mathbf{u}}_2]\underline{\mathbf{\Lambda}} & \\ \Rightarrow & \underline{\mathbf{J}}\,\underline{\mathbf{u}}_i & = & \underline{\lambda}_i\,\underline{\mathbf{u}}_i & \quad i = \{1, 2\}. \end{array} \tag{5}$$

This eigenvalue decomposition requires a closer look.

2.1. Mathematical View

The complex eigenvalue equation has an often ignored but interesting property which will help to understand the behavior of the coneigenvalue equation in the next section. With the eigenvector $\underline{\mathbf{u}}_i = \underline{\mathbf{v}}_i\, e^{j\varphi_i}$ follows

$$\begin{array}{rlcl} & \underline{\mathbf{J}}\,\underline{\mathbf{v}}_i\, e^{j\varphi_i} & = & \underline{\lambda}_i\,\underline{\mathbf{v}}_i\, e^{j\varphi_i} \\ \Rightarrow & \underline{\mathbf{J}}\,\underline{\mathbf{v}}_i & = & \underline{\lambda}_i\,\underline{\mathbf{v}}_i \end{array} \tag{6}$$

This shows that the eigenvalue equation is fulfilled by the tuples $(\underline{\lambda}_i,\, \underline{\mathbf{u}}_i\, e^{j\varphi_i})$ with $\varphi_i \in [0, 2\pi)$. Due to the arbitrary phase factor of the eigenvector there exists an infinite number of allowed eigenvectors belonging to each eigenvalue.

The same property can be observed for (4): It is possible to define $\underline{\mathbf{U}} = \underline{\mathbf{V}}\,\underline{\mathbf{\Phi}}$ where

$$\underline{\mathbf{\Phi}} = \begin{bmatrix} e^{j\varphi_1} & 0 \\ 0 & e^{j\varphi_2} \end{bmatrix}. \tag{7}$$

When inserted into (4) this yields

$$\begin{array}{rlcl} & (\underline{\mathbf{V}}\,\underline{\mathbf{\Phi}})^H\,\underline{\mathbf{J}}\,\underline{\mathbf{V}}\,\underline{\mathbf{\Phi}} & = & \underline{\mathbf{\Lambda}} \\ \Rightarrow & \underline{\mathbf{V}}^H\,\underline{\mathbf{J}}\,\underline{\mathbf{V}} & = & \underline{\mathbf{\Phi}}\,\underline{\mathbf{\Lambda}}\underline{\mathbf{\Phi}}^H \\ \Rightarrow & \underline{\mathbf{V}}^H\,\underline{\mathbf{J}}\,\underline{\mathbf{V}} & = & \underline{\mathbf{\Lambda}} \end{array} \tag{8}$$

Thus also the phase factors of the column vectors in $\underline{\mathbf{U}}$ can be chosen arbitrarily without influencing the diagonal matrix $\underline{\mathbf{\Lambda}}$.

2.2. Physical Interpretation

The Jones matrix describes the connection between an electromagnetic wave approaching a target $\underline{\mathbf{E}}_i^+$ and an electromagnetic wave scattered by a target $\underline{\mathbf{E}}_s^+$ with $\underline{\mathbf{E}}_s^+ = \underline{\mathbf{J}}\,\underline{\mathbf{E}}_i^+$. Here the directional Jones vector notation by Graves and Lüneburg is used to differentiate between different propagation directions of the wave [5, 6]. Since $\underline{\mathbf{E}}_i^+$ and $\underline{\mathbf{E}}_s^+$ both belong to the polarization space $\mathcal{P}^+$, the waves have the same direction of propagation.

Figure 1 is a visualization of the derivation given in (8). Without loss of generality a forward scattering case is depicted which allows an easier interpretation of the shown effect. The targets are represented by the diagonal matrix $\underline{\mathbf{\Lambda}}$ and depicted as clouds in the middle of the figure. Moreover, the wave vectors of incident and scattered electromagnetic waves are shown as horizontal black arrows and the polarization plane of each wave is indicated by a gray vertical line. Wave vector and polarization plane together describe the reference coordinate system of each wave.

The upper part of the figure can be interpreted as follows: At two certain points in space the coordinate systems of the incident and scattered electromagnetic waves $\underline{\mathbf{E}}_i^+$ and $\underline{\mathbf{E}}_s^+$ are located. $\underline{\mathbf{J}}$ can be transformed with a similarity transformation to a diagonal matrix $\underline{\mathbf{\Lambda}}$. This diagonal matrix describes the same target, but by using the electromagnetic waves $\hat{\underline{\mathbf{E}}}_i^+$ and $\hat{\underline{\mathbf{E}}}_s^+$ which have a new polarization basis due to the similarity transformation. This change of the polarization basis is conducted by the unitary matrix $\underline{\mathbf{U}}$. It is important to understand that $\underline{\mathbf{U}}$ not only changes the basis vectors in the polarization plane, but can also change the phase factors of the column vectors and therefore the location of the coordinate systems of the electromagnetic waves. Therefore $\hat{\underline{\mathbf{E}}}_i^+$ and $\hat{\underline{\mathbf{E}}}_s^+$ are drawn directly before and behind the target.

The lower part of Fig. 1 shows what happens if $\underline{\mathbf{U}}$ is replaced by $\underline{\mathbf{U}} = \underline{\mathbf{V}}\,\underline{\mathbf{\Phi}}$: Since $\underline{\mathbf{\Phi}}$ contains phase factors only, the matrix $\underline{\mathbf{V}}$ does not change the basis vectors in the polarization plane, but

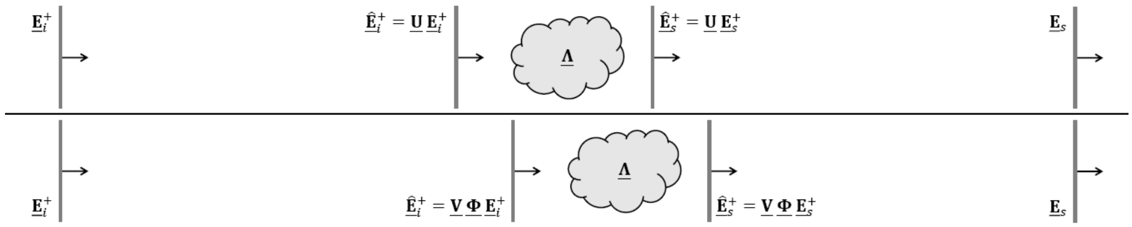

Figure 1: Decomposition of the Jones matrix without (top) and with (bottom) substitution of $\underline{\mathbf{U}} = \underline{\mathbf{V}}\,\underline{\mathbf{\Phi}}$.

the location of the reference coordinate systems. $\hat{\underline{\mathbf{E}}}_i^+$, $\hat{\underline{\mathbf{E}}}_s^+$ and $\underline{\mathbf{\Lambda}}$ are shifted uniformly to a different position along the same direction. This means that $\underline{\mathbf{\Phi}}$ shifts the target position without changing the entries of $\underline{\mathbf{\Lambda}}$. This behavior is well-known for forward scattering: Radar systems exploiting forward scattering are not able to determine the location of a target since changing the position of the target does not introduce a phase change between $\underline{\mathbf{E}}_i^+$ and $\underline{\mathbf{E}}_s^+$ [7]. It is exactly this behavior which leads to the indeterminacy of the phase factor of the eigenvectors and unambiguously defined eigenvalues.

3. CONDIAGONALIZATION OF SINCLAIR MATRICES

If the Sinclair matrix $\underline{\mathbf{S}}$ is symmetric, then it is consimilar to the diagonal matrix $\underline{\mathbf{\Sigma}}$ with [5]:

$$\underline{\mathbf{U}}^T\,\underline{\mathbf{S}}\,\underline{\mathbf{U}} = \underline{\mathbf{\Sigma}} \tag{9}$$

Using $\underline{\mathbf{U}} = [\underline{\mathbf{u}}_1\ \ \underline{\mathbf{u}}_2]$ Equation (9) can be rewritten to:

$$\begin{array}{rlcll} & \underline{\mathbf{S}}\,\underline{\mathbf{U}} & = & \underline{\mathbf{U}}^*\,\underline{\mathbf{\Sigma}} & \\ \Rightarrow & \underline{\mathbf{S}}[\underline{\mathbf{u}}_1\ \ \underline{\mathbf{u}}_2] & = & [\underline{\mathbf{u}}_1^*\ \ \underline{\mathbf{u}}_2^*]\underline{\mathbf{\Sigma}} & \\ \Rightarrow & \underline{\mathbf{S}}\,\underline{\mathbf{u}}_i & = & \underline{\sigma}_i\,\underline{\mathbf{u}}_i^* & \quad i = \{1, 2\}. \end{array} \tag{10}$$

The last row corresponds to the coneigenvalue equation. Thus the diagonal entries of $\underline{\mathbf{\Sigma}}$ are coneigenvalues. It should be noted that the notation $\underline{\mathbf{S}}\,\underline{\mathbf{u}}_i^* = \underline{\sigma}_i\,\underline{\mathbf{u}}_i$ is more frequently used in literature [8], but is equal to (10) by defining $\underline{\mathbf{U}}$ to $\underline{\mathbf{U}} = [\underline{\mathbf{u}}_1^*\ \ \underline{\mathbf{u}}_2^*]$ instead [6].

3.1. Mathematical View

With the eigenvector $\underline{\mathbf{u}}_i = \underline{\mathbf{v}}_i\,e^{j\varphi_i}$ follows

$$\begin{array}{rll} & \underline{\mathbf{S}}\,\underline{\mathbf{v}}_i\,e^{j\varphi_i} & = \underline{\sigma}_i\,\left(\underline{\mathbf{v}}_i\,e^{j\varphi_i}\right)^* \\ & \underline{\mathbf{S}}\,\underline{\mathbf{v}}_i\,e^{j\varphi_i} & = \underline{\sigma}_i\,\underline{\mathbf{v}}_i^*\,e^{-j\varphi_i} \\ \Rightarrow & \underline{\mathbf{S}}\,\underline{\mathbf{v}}_i & = \underline{\sigma}_i\,\underline{\mathbf{v}}_i^*\,e^{-j2\varphi_i} \\ \Rightarrow & \underline{\mathbf{S}}\,\underline{\mathbf{v}}_i & = \underline{\rho}_i\,\underline{\mathbf{v}}_i^* \end{array} \tag{11}$$

where $\underline{\rho}_i = \underline{\sigma}_i\,e^{-j2\varphi_i}$. Since $\varphi_i \in [0, 2\pi)$ the phase of the coneigenvalue can be chosen arbitrarily. This derivation can be often found in literature [5, 6, 8] and led Lüneburg to the conclusion recapitulated in the introduction. Although it is correct that the phase of the coneigenvalues is indetermined, the role of the coneigenvectors should not be ignored. The coneigenvalue equation is fulfilled by the tuples $(\underline{\sigma}_i\,e^{-j2\varphi_i},\ \underline{\mathbf{u}}_i\,e^{j\varphi_i})$ with $\varphi_i \in [0, 2\pi)$. Thus there exists an infinite number of tuples and therefore an infinite number of coneigenvectors and coneigenvalues, but it can also be seen that there belongs exactly one coneigenvector to each coneigenvalue. In contrast to the eigenvalue equation where eigenvectors with arbitrary phase factors are permitted, the coneigenvalue equation can only be fulfilled for a distinct coneigenvector-coneigenvalue pair. This also means that if the coneigenvalue equation is solved in a way which neglects the phase of the coneigenvalues, this phase information is not lost but "shifted" into the coneigenvectors. This is of special importance for Huynens skip angle: It now becomes clear that both the phase information of the coneigenvalues and the phase factor of the coneigenvectors are required to reconstruct the skip angle completely. Basically this was already considered in the Equations (2) and (3) taken from [1] since there the two angles α and ν were introduced, where α was used to describe the part of the skip angle which can be found within the coneigenvector and ν being the part found in the coneigenvalue. Unfortunately in [1] only the angle ν was identified as skip angle. From the explanations given above it follows that the skip angle must be defined to $\hat{\nu} = \alpha - \nu$.

Again, Equation (9) should be considered. By defining $\underline{\mathbf{U}} = \underline{\mathbf{V}}\,\underline{\mathbf{\Phi}}$, where $\underline{\mathbf{\Phi}}$ is described in (7), it follows

$$\begin{array}{rll} & (\underline{\mathbf{V}}\,\underline{\mathbf{\Phi}})^T\,\underline{\mathbf{S}}\,\underline{\mathbf{V}}\,\underline{\mathbf{\Phi}} & = \underline{\mathbf{\Sigma}} \\ \Rightarrow & \underline{\mathbf{V}}^T\,\underline{\mathbf{S}}\,\underline{\mathbf{V}} & = \underline{\mathbf{\Phi}}^*\,\underline{\mathbf{\Sigma}}\underline{\mathbf{\Phi}}^* \\ \Rightarrow & \underline{\mathbf{V}}^T\,\underline{\mathbf{S}}\,\underline{\mathbf{V}} & = \underline{\mathbf{\Sigma}}\,(\underline{\mathbf{\Phi}}^*)^2 \\ \Rightarrow & \underline{\mathbf{V}}^T\,\underline{\mathbf{S}}\,\underline{\mathbf{V}} & = \underline{\mathbf{P}} \end{array} \tag{12}$$

with $\underline{\mathbf{P}} = \underline{\mathbf{\Sigma}}\,(\underline{\mathbf{\Phi}}^*)^2$. Thus although the phases of the diagonal entries of $\underline{\mathbf{\Sigma}}$ are indetermined, they dictate the phase factors of the column vectors in $\underline{\mathbf{U}}$.

3.2. Physical Interpretation

The Sinclair matrix describes the connection between an electromagnetic wave approaching a target $\underline{\mathbf{E}}_i^+$ and an electromagnetic wave scattered by a target $\underline{\mathbf{E}}_s^-$ with $\underline{\mathbf{E}}_s^- = \underline{\mathbf{S}}\,\underline{\mathbf{E}}_i^+$. Again the directional Jones vector notation helps to differentiate between different propagation directions of the wave. $\underline{\mathbf{E}}_i^+$ belongs to the polarization space $\mathcal{P}^+$, whereas $\underline{\mathbf{E}}_s^-$ belongs to the polarization space $\mathcal{P}^-$ [6]. Thus the waves propagate in different directions.

Figure 2 shows a similar scenario as Fig. 1, but this time backward scattering is depicted. Due to the different polarization spaces of $\underline{\mathbf{E}}_i^+$ and $\underline{\mathbf{E}}_s^-$ the wave vectors point in the same direction, forming an antenna-oriented reference system. By replacing $\underline{\mathbf{U}}$ with $\underline{\mathbf{U}} = \underline{\mathbf{V}}\,\underline{\mathbf{\Phi}}$, the coordinate systems of $\hat{\underline{\mathbf{E}}}_i^+$ and $\hat{\underline{\mathbf{E}}}_s^-$ both shift into the direction of $\underline{\mathbf{E}}_i^+$ and $\underline{\mathbf{E}}_s^-$. Thus $\hat{\underline{\mathbf{E}}}_i^+$ and $\hat{\underline{\mathbf{E}}}_s^-$ depart from $\underline{\mathbf{\Sigma}}$ and by this change the diagonal matrix to $\underline{\mathbf{P}} = \underline{\mathbf{\Sigma}}\,(\underline{\mathbf{\Phi}}^*)^2$. This was not the case in Fig. 1: There $\hat{\underline{\mathbf{E}}}_i^+$ shifted away from $\underline{\mathbf{E}}_i^+$ and $\hat{\underline{\mathbf{E}}}_s^+$ came closer to $\underline{\mathbf{E}}_s^+$, leaving $\underline{\mathbf{\Lambda}}$ unchanged.

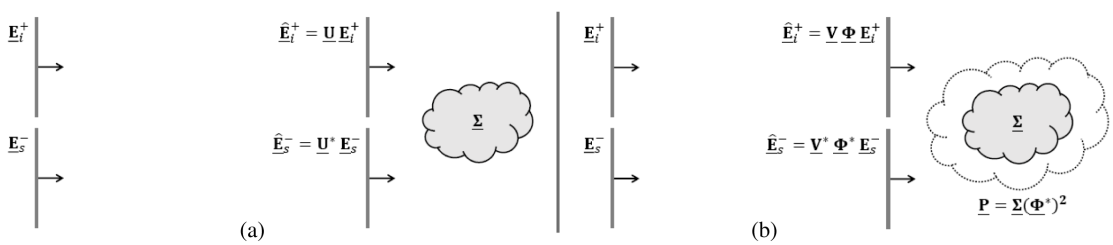

Figure 2: Decomposition of the Sinclair matrix (a) without and (b) with substitution of $\underline{\mathbf{U}} = \underline{\mathbf{V}}\,\underline{\mathbf{\Phi}}$.

The displacement of $\hat{\underline{\mathbf{E}}}_i^+$ and $\hat{\underline{\mathbf{E}}}_s^-$ is accompanied by a change of the target $\underline{\mathbf{\Sigma}}$ to the target $\underline{\mathbf{P}}$. It is important to note that only the phases of the diagonal entries of $\underline{\mathbf{\Sigma}}$ and $\underline{\mathbf{P}}$ differ. This is indicated in Fig. 2 by the dotted cloud surrounding the original $\underline{\mathbf{\Sigma}}$: The target appears to be closer to $\underline{\mathbf{E}}_i^+$ and $\underline{\mathbf{E}}_s^-$, but is also "bloated" by the additional phase factors $\underline{\mathbf{\Phi}}$ in $\underline{\mathbf{P}}$. A similar mechanism is known for corrugated surfaces which can be used to change the surface impedance by introducing a phase shift between the actual metallic surface and the artificial surface realized by the corrugations [9]. Thus for a coneigenvalue decomposition the phase shift between $\underline{\mathbf{\Sigma}}$ and $\underline{\mathbf{P}}$ can be understood as a change of the surface impedance of the target to compensate the shift of the reference coordinate system caused by a variation of the coneigenvector's phase factor.

3.3. The Role of the Skip Angle

The physical interpretation given in the last section leads to the question if an unambiguous determination of the skip angle can be made possible in realistic scenarios. Absolute phase factors of Jones vectors are normally difficult to handle since they do not only describe the target's position but also incorporate effects of the target and the radar system itself [2]. Therefore it seems to be challenging to extract the required skip angle from $\underline{\mathbf{V}}$ instead from $\underline{\mathbf{\Sigma}}$ as motivated in Section 3.1. However, the explanation in the last section did not distinguish between the two orthogonal components of the polarization basis. A spatial displacement will change the phase of both Jones vector components by the same amount. This fact was used to explain the connection between spatial displacement and a surface impedance change of the target. The matrix $\underline{\mathbf{\Phi}}$ has one more degree of freedom since the phases of its diagonal entries can be set to independent values φ_1 and φ_2. The decomposition

$$\underline{\mathbf{\Phi}} = \begin{bmatrix} e^{j\varphi_1} & 0 \\ 0 & e^{j\varphi_2} \end{bmatrix} = e^{j\xi'} \begin{bmatrix} e^{+j\alpha'} & 0 \\ 0 & e^{-j\alpha'} \end{bmatrix} \tag{13}$$

results in the absolute phase angle ξ' which describes the aforementioned spatial displacement and the angle α' which can change the phases of the Jones vector components in antipodal directions and therefore is not suitable to describe a spatial displacement. Therefore the behavior described by α' must be a property of the target. A comparison to (3) shows that α' has the same form as the matrix $\underline{\mathbf{U}}_\alpha$ and therefore is (at least a part of) the skip angle.

4. CONCLUSION

The discussion conducted in this paper leads to the following insights:

- A coneigenvalue equation can be fulfilled by inifinitely many coneigenvalues, but each coneigenvalue is linked to a single coneigenvector which is required to fulfill the coneigenvalue equation. In contrast to this, an eigenvalue equation can be fulfilled by only one eigenvalue, but there are infinitely many eigenvectors which can be used together with the eigenvalue to fulfill the eigenvalue equation.
- Due to this the skip angle can only be reconstructed from the Sinclair matrix if the phase information of both the coneigenvectors and the coneigenvalues is taken into account.
- For a target represented by a Sinclair matrix it is not possible to differentiate between a target at a certain distance and a target with altered surface impedance at a closer distance. For a target represented by a Jones matrix it is not possible to distinguish targets at different positions.
- The skip angle is a target property and is not affected by the target's position. It can be reconstructed from the Sinclair matrix by an arbitrary coneigenvalue-coneigenvector pair.

This shows that the Huynen-Euler parameterization requires coneigenvalue decompositions which return both the coneigenvalues and the coneigenvectors. This is not always the case; a decomposition utilizing the Graves matrix will return real-valued coneigenvalues but not the associated coneigenvectors for example [1]. Therefore further investigations in this field of research are required to find appropriate and numerically stable coneigenvalue decompositions.

REFERENCES

1. Lee, J. and E. Pottier, *Polarimetric Radar Imaging: From Basics to Applications*, CRC Press, Boca Raton, 2009.
2. Huynen, J. R., *Phenomenological Theory of Radar Targets*, Ph.D. dissertation, TU Delft, 1970.
3. Dallmann, T. and D. Heberling, "Discrimination of scattering mechanisms via polarimetric rcs imaging [measurements corner]," *Antennas and Propagation Magazine, IEEE*, Vol. 56, No. 3, 154–165, 2014.
4. Baird, C. S., "Design and analysis of an Euler transformation algorithm applied to full-polarimetric ISAR imagery," Ph.D. dissertation, University of Massachusetts Lowell, Massachusetts, 2007.
5. Lüneburg, E., "Aspects of radar polarimetry," *Turk J. Elec. Engin.*, Vol. 10, No. 2, 219–243, 2002.
6. Serbest, A. H. and S. Cloude, *Direct and Inverse Electromagnetic Scattering*, CRC Press, Boca Raton, 1996.
7. Willis, N. J., *Bistatic Radar*, SciTech Publishing, Raleigh, 2005.
8. Horn, R. A. and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 2013.
9. Harrington, R. F., *Time-Harmonic Electromagnetic Fields*, Wiley-IEEE Press, New York, 2001.