A Necessary Condition for Application of Topological Derivative in Limited-aperture Inverse Scattering Problem

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Abstract— Various studies have shown that the topological derivative-based non-iterative imaging algorithm can be employed in limited-aperture inverse scattering problems. However, this fact has been verified through the results of numerical simulations, and only a sufficient condition for its application has been theoretically discovered. In this work, we consider the topological derivative for the imaging of a thin, crack-like dielectric inhomogeneity, and explore a necessary condition for its application in a limited-aperture inverse scattering problem. This is based on a relationship between topological derivative imaging functions and infinite series of Bessel functions of integer order of the first kind. This relationship implies that the necessary condition is highly dependent on the shape of the unknown inhomogeneity. Numerical results are presented in order to support our findings.

1. INTRODUCTION

In this study, we consider an inverse scattering problem for discovering the shape of unknown thin curve-like dielectric inhomogeneities from measured boundary data. In past research, various algorithms for identifying the shape of unknown objects have been suggested. Most of these algorithms are based on Newton-type iteration schemes. However, in order to induce a good result, these schemes must start with a good initial guess, which is close to the unknown object. In order to produce a good initial guess, alternative non-iterative algorithms have been developed.

Among them, a topological derivative strategy has been applied with success. This strategy has been considered for shape optimization problems [6, 8, 9, 15], and was then successfully combined with the level-set method (see [2, 7, 12, 14]). Surprisingly, some remarkable studies [4, 5] found that topological derivatives present a non-iterative imaging algorithm. However, the majority of research has considered full-aperture problems. For more details, see [3, 11, 13]. Though it is true that various studies have confirmed that topological derivatives can be applied in limited-aperture problems, but this fact has been dependent on simulation results. Therefore, an analysis of topological derivatives in limited-aperture problems is an important requirement. Motivated by this fact, the mathematical structure of topological derivatives in limited-aperture inverse scattering problems has been established, in [1]. In this interesting research, a sufficient condition for its application has been identified theoretically, but a necessary condition remains unknown. Therefore, in this paper, we explore a necessary condition for the application of topological derivatives in limited-aperture inverse scattering problems. This is based on a relationship between topological derivatives and infinite series of Bessel functions of the first kind of the integer order.

This paper is organized as follows. In Section 2, we briefly introduce the two-dimensional direct scattering problem and the topological derivative imaging algorithm. In Section 3, we deduce a necessary condition for a successful application of topological derivatives in limited-aperture inverse scattering problems. In Section 4, several results of numerical simulations with noisy data are presented in order to support our findings. A brief conclusion is given in Section 5.

2. DIRECT SCATTERING PROBLEMS AND TOPOLOGICAL DERIVATIVES

Let $\Omega \subset \mathbb{R}^2$ be a homogeneous domain with a smooth boundary $\partial\Omega$, which contains a homogeneous thin inhomogeneity Γ with a small thickness $2h$. That is,

$$\Gamma = \{\mathbf{x} + \eta\mathbf{n}(\mathbf{x}) : \mathbf{x} \in \sigma, -h \leq \eta \leq h\},$$

where $\mathbf{n}(\mathbf{x})$ is the unit normal to σ at x, and σ denotes a simple, smooth curve in $\mathbb{R}^2$ describing the supporting curve of Γ. In this study, we assume that the applied angular frequency is of the form $\omega = 2\pi\lambda$, where λ is a given wavelength and satisfying $h \ll \lambda$. All materials involved are non-magnetic (permeability $\mu \equiv 1$), and characterized by their dielectric permittivity at the frequency of operation ω. We define the piecewise constant permittivity $\varepsilon(\mathbf{x})$ as

$$\varepsilon(\mathbf{x}) := \begin{cases} \varepsilon_0 & \text{for } \mathbf{x} \in \Omega \backslash \bar{\Gamma} \\ \varepsilon_\star & \text{for } \mathbf{x} \in \Gamma \end{cases}$$

Let $u_{\mathrm{T}}^{(n)}(\mathbf{x};\omega)$ be the time-harmonic total field, satisfying the boundary value problem

$$\begin{cases} \triangle u_{\mathrm{T}}^{(n)}(\mathbf{x};\omega) + \omega^2 \varepsilon(\mathbf{x}) u_{\mathrm{T}}^{(n)}(\mathbf{x};\omega) = 0 & \text{for } \mathbf{x} \in \Omega, \\ \dfrac{\partial u_{\mathrm{T}}^{(n)}(\mathbf{x};\omega)}{\partial \boldsymbol{\nu}(\mathbf{x})} = \dfrac{\partial \exp(i\omega \boldsymbol{\theta}_n \cdot \mathbf{x})}{\partial \boldsymbol{\nu}(\mathbf{x})} & \text{for } \mathbf{x} \in \partial\Omega, \end{cases} \tag{1}$$

with transmission conditions on the boundary of Γ. Here, $\boldsymbol{\theta}_n = [\cos(\theta_n), \sin(\theta_n)]$ denotes a two-dimensional vector on the connected, proper subset of the unit circle $\mathbb{S}^1$, such that

$$\theta_n = \theta_1 + \frac{n-1}{N-1}(\theta_N - \theta_1), \quad 0 < \theta_N - \theta_1 < 2\pi.$$

Similarly, let $u_{\mathrm{B}}^{(n)}(\mathbf{x};\omega) = \exp(i\omega\boldsymbol{\theta}_n \cdot \mathbf{x})$ be the background solution of (1). In this paper, we assume that $u_{\mathrm{T}}^{(n)}(\mathbf{x};\omega)$ and $u_{\mathrm{B}}^{(n)}(\mathbf{x};\omega)$ are measured everywhere on $\partial\Omega$. Based on this, we can define an energy functional:

$$\mathbb{E}(\Omega;\omega) := \frac{1}{2}\sum_{n=1}^{N} \int_{\partial\Omega} \left| u_{\mathrm{T}}^{(n)}(\mathbf{x};\omega) - u_{\mathrm{B}}^{(n)}(\mathbf{x};\omega) \right|^2 dS(\mathbf{x}). \tag{2}$$

In order to consider the topological derivative, we create a dielectric inhomogeneity, say Σ, of small diameter r at $\mathbf{z} \in \Omega \backslash \partial\Omega$, and denote corresponding domain as $\Omega|\Sigma$. Then, the topological derivative $d_T\mathbb{E}(\mathbf{z};\omega)$ can be defined as follows (see [13])

$$d_T\mathbb{E}(\mathbf{z};\omega) = \lim_{r \to 0+} \frac{\mathbb{E}\left(\Omega|\Sigma;\omega\right) - \mathbb{E}\left(\Omega;\omega\right)}{\varphi(r;\omega)}, \tag{3}$$

where $\varphi(r;\omega) \longrightarrow 0$ as $r \longrightarrow 0+$. From (3), we can obtain the asymptotic expansion:

$$\mathbb{E}(\Omega|\Sigma;\omega) = \mathbb{E}(\Omega;\omega) + \varphi(r;\omega) d_T\mathbb{E}(\mathbf{z};\omega) + o(\varphi(r;\omega)). \tag{4}$$

Then, $d_T\mathbb{E}(\mathbf{z};\omega)$ can be represented as

$$d_T\mathbb{E}(\mathbf{z};\omega) = \operatorname{Re}\sum_{n=1}^{N}\left(u_{\mathrm{A}}^{(n)}(\mathbf{z};\omega) \overline{u_{\mathrm{B}}^{(n)}(\mathbf{z};\omega)} \right),$$

where $u_{\mathrm{A}}^{(n)}(\mathbf{x};\omega)$ satisfies the following adjoint problem:

$$\begin{cases} \Delta u_{\mathrm{A}}^{(n)}(\mathbf{x};\omega) + \omega^2 u_{\mathrm{A}}^{(n)}(\mathbf{x};\omega) = 0 & \text{for } \mathbf{x} \in \Omega, \\ \dfrac{\partial u_{\mathrm{A}}^{(n)}(\mathbf{x};\omega)}{\partial \boldsymbol{\nu}(\mathbf{x})} = u_{\mathrm{T}}^{(n)}(\mathbf{x};\omega) - u_{\mathrm{B}}^{(n)}(\mathbf{x};\omega) & \text{for } \mathbf{x} \in \partial\Omega. \end{cases} \tag{5}$$

Then, the structure of $d_T\mathbb{E}(\mathbf{z};\omega)$ can be written as follows.

Lemma 2.1 (See [11, 13]). *Suppose that N and ω are sufficiently large, then*

$$d_T\mathbb{E}(\mathbf{z};\omega) \approx \int_{\sigma} (\varepsilon_\star - \varepsilon_0) \exp(i\boldsymbol{\theta}_n \cdot (\mathbf{x} - \mathbf{z})) d\sigma(\mathbf{x}). \tag{6}$$

Note that, in limited-aperture problem, (6) can be represented as follows. This result plays an important role in establishing the necessary condition for application derived in the next section.

Lemma 2.2 (See [1]). *Let* $\boldsymbol{\theta}_n = [\cos(\theta_n), \sin(\theta_n)]$ *and* $\mathbf{x} - \mathbf{z} = \rho[\cos(\phi), \sin(\phi)]$. *If* N *and* ω *are sufficiently large, then*

$$d_T\mathbb{E}(\mathbf{z};\omega) \approx \int_\sigma (\varepsilon_\star - \varepsilon_0)\left(J_0(\omega|\mathbf{x}-\mathbf{z}|) + \frac{\mathbb{D}(\omega|\mathbf{x}-\mathbf{z}|, \theta_1, \theta_N)}{\theta_N - \theta_1}\right) d\sigma(\mathbf{x}). \tag{7}$$

Here,

$$\mathbb{D}(\omega|\mathbf{x}-\mathbf{z}|, \theta_1, \theta_N) = 2\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin\{m(\theta_N - \theta_1)\}\cos\{m(\theta_N + \theta_1 - 2\phi)\} J_{2m}(\omega|\mathbf{x}-\mathbf{z}|).$$

3. NECESSARY CONDITION OF TOPOLOGICAL DERIVATIVES

Based on the structure of $d_T\mathbb{E}(\mathbf{z};\omega)$ in (7), we observe that the term $J_0(\omega|\mathbf{x}-\mathbf{z}|)$ contributes to the imaging performance, while $\mathbb{D}(\omega|\mathbf{x}-\mathbf{z}|, \theta_1, \theta_N)$ is a disturbance. Therefore, eliminating the term $\mathbb{D}(\omega|\mathbf{x}-\mathbf{z}|, \theta_1, \theta_N)$ will guarantee good results, i.e., one must find some conditions such that

$$\sum_{m=1}^{\infty} \sin\{m(\theta_N - \theta_1)\}\cos\{m(\theta_N + \theta_1 - 2\phi)\} J_{2m}(\omega|\mathbf{x}-\mathbf{z}|) \equiv 0$$

for all $\mathbf{z} \in \Omega$. If $\omega = +\infty$, we can expect a good result, but this is an ideal condition. Therefore, let us assume that $\omega < +\infty$. Because $\mathbf{z}$ is arbitrary, we cannot control the value of the term $J_{2n}(\omega|\mathbf{x}-\mathbf{z}|)$. Therefore, we must find a condition on the range of incident directions, such that for all $m \in \mathbb{N}$,

$$\sin\{m(\theta_N - \theta_1)\}\cos\{m(\theta_N + \theta_1 - 2\phi)\} \equiv 0.$$

A simple way is to select θ_1 and θ_N such that $\theta_N - \theta_1 = \pi/2$ and $\theta_N + \theta_1 - 2\phi = \pi/2$. That is, if $\theta_1 = \phi$ and $\theta_N = \phi + \pi/2$, good results can be obtained via topological derivatives.

Based on the above observation, we also observe the following. Let $\phi(\mathbf{x})$ denote the slope of the tangential line at $\mathbf{x} \in \sigma$. Then, in order to obtain good results, θ_1 and θ_N must satisfy

$$\theta_1 = \min\{\phi(\mathbf{x}) : \mathbf{x} \in \sigma\} \quad \text{and} \quad \theta_N = \max\{\phi(\mathbf{x}) : \mathbf{x} \in \sigma\} + \frac{\pi}{2}. \tag{8}$$

This is a necessary condition for the application of topological derivatives in limited-aperture problems. Note that, based on [1], a sufficient condition for application is that $\theta_1 = 0$ and $\theta_N = \pi$. On the basis of these conditions, we conclude that the necessary condition is highly dependent on the shape of the thin inclusion, while the sufficient condition is not.

Remark 3.1. *In recent work [1, 10, 11], it has been confirmed that the application of multi-frequencies guarantees better a imaging performance than the application of a single frequency. Therefore, we consider the following normalized multi-frequency topological derivative: for several frequencies* $\{\omega_f : f = 1, 2, \ldots, F\}$, *define*

$$\mathbb{M}(\mathbf{z}; F) := \frac{1}{F}\sum_{f=1}^{F} \frac{d_T\mathbb{E}(\mathbf{z};\omega_f)}{\max[d_T\mathbb{E}(\mathbf{z};\omega_f)]}. \tag{9}$$

4. RESULTS OF NUMERICAL SIMULATION

In this section, some numerical results are presented in order to support our observation from the previous section. For this purpose, three σ_j characteristics of the thin inhomogeneity Γ_j are chosen for our illustration:

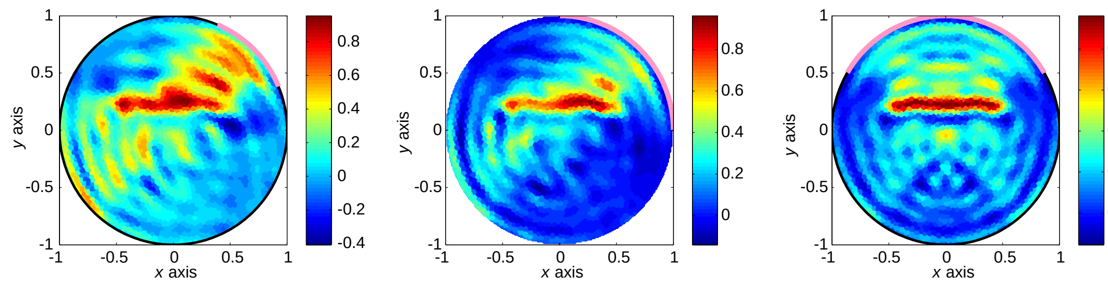
$$\begin{aligned}
\sigma_1 &= \{[s, 0.2] : -0.5 \le s \le 0.5\} \\
\sigma_2 &= \left\{\left[s + 0.2, s^3 + s^2 - 0.5\right] : -0.5 \le s \le 0.5\right\} \\
\sigma_3 &= \left\{\left[s, 0.5s^2 + 0.1\sin(3\pi(s + 0.7))\right] : -0.7 \le s \le 0.7\right\}.
\end{aligned}$$

Throughout this section, we denote be the permittivities of Γ_j by ε_j, and their values are equal to 5. The thickness h of the Γ_j are equally set to 0.015, and ε_0 is set as 1. The applied frequency is $\omega_f = 2\pi/\lambda_f$, where $f = 1, 2, \cdots, 10$ is the given wavelength. In this paper, $N = 16$ incident directions are to applied λ_f, and are equi-distributed within the interval $[\lambda_{10}, \lambda_1] = [0.3, 0.6]$. In order to demonstrate robustness, a white Gaussian noise with 20 dB signal-to-noise ratio (SNR) is added to the unperturbed boundary data $u_{\mathrm{T}}^{(n)}(\mathbf{x}; \omega)$ via a standard MATLAB command `awgn`.

Maps of $\mathbb{M}(\mathbf{z}; 10)$ with various ranges of directions for Γ_1 are presented in Figure 1. Note that if the range of directions is narrow, one cannot identify the shape of Γ_1, because the disturbing term $\mathbb{D}(\omega|\mathbf{x} - \mathbf{z}|, \theta_1, \theta_N)$ still exists. See Figure 1(a). Note that for Γ_1, we can easily observe that $\phi(\mathbf{x}) \equiv 0$. Therefore, based on (8), the selection of $\theta_1 = 0$ and $\theta_N = \pi/2$ will guarantee a good result. See Figure 1(b). In addition, it is true that a wider range of directions yields better results (see Figure 1(c)).

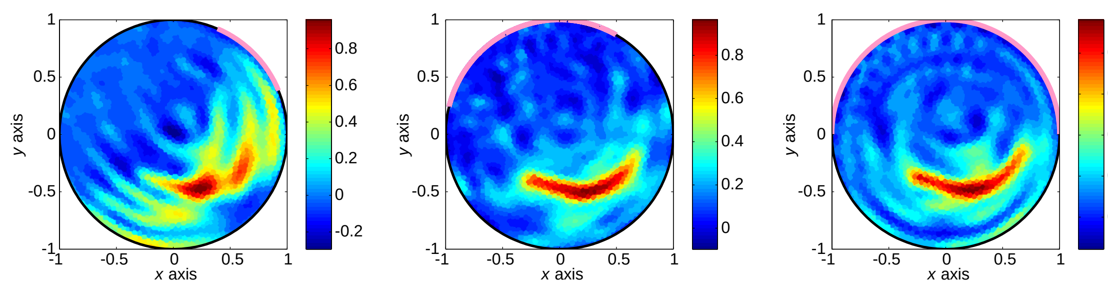
For a curve-like thin inhomogeneity Γ_2, as it is no longer straight-line shaped, $\phi(\mathbf{x})$ is no longer constant. Therefore, a selection of θ_1 and θ_N satisfying (8) will guarantee a good result. See Figure 2(b). Similar to the results in Figure 1, a result with poor resolution occurs when the range of directions is narrow, while one can obtain a good result when the range is sufficiently large. See Figures 2(a) and 2(c).

For the final example, let us consider the imaging of a thin oscillating inhomogeneity Γ_3. Based on the results of Figure 3, we observe that the necessary condition must be the same as the sufficient condition for obtaining an acceptable image of an oscillating inhomogeneity.



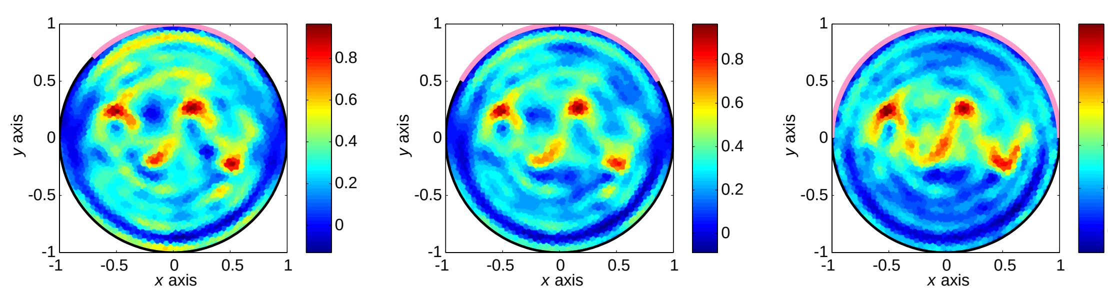
(a) $\theta_1=\pi/8$ and $\theta_N=3\pi/8$ (b) θ_1 and θ_N satisfy (8) (c) $\theta_1=\pi/6$ and $\theta_N=5\pi/6$

Figure 1: Maps of $\mathbb{M}(\mathbf{z}; 10)$ when the thin dielectric inhomogeneity is Γ_1.



(a) $\theta_1=\pi/8$ and $\theta_N=3\pi/8$ (b) θ_1 and θ_N satisfy (8) (c) $\theta_1=0$ and $\theta_N=\pi$

Figure 2: Similar to Figure 1, except that the inhomogeneity is Γ_2.



(a) $\theta_1=\pi/4$ and $\theta_N=3\pi/4$ (b) $\theta_1=\pi/6$ and $\theta_N=5\pi/6$ (c) $\theta_1=0$ and $\theta_N=\pi$

Figure 3: Similar to Figure 1, except that the inhomogeneity is Γ_3.

5. CONCLUDING REMARKS

In this contribution, we have considered the topological derivative approach to the non-iterative imaging of a thin dielectric inhomogeneity, in a limited-aperture inverse scattering problem. We reviewed a relationship with infinite series of Bessel functions of integer order of the first kind, and discovered a necessary condition for successful application to limited-aperture problems. We presented the results of some numerical simulations, which show that the discovered necessary condition is valid for the imaging of a thin inclusion, but is restricted in the imaging of an oscillating inhomogeneity. Note that the result obtained in this contribution does not guarantee the true shape of inhomogeneity. However, fortunately, they can provide a good initial guess for an iterative based algorithm or the level-set method. For more details, refer to [2, 6, 7, 12, 14].

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