

# A Generalized Vector-potential Integral Formulation for the Paraboloidal Reflector Antenna

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**Abstract**— This paper develops a generalized vector magnetic potential integral formulation for the paraboloidal reflector antenna, using the elliptic paraboloid geometry as basis. First, parametric expressions informed by the problem geometry are specified for the conventional ‘field’ and ‘source’ points, and the ellipse’s major and minor axes are described by a common expression, also based on the geometry. When the expressions are utilized in the integral equation for the vector magnetic potential for the problem, and following the usual ‘magnitude’ and ‘phase’ approximations, an expression for the radiation field, which may be described as a general expression for certain special cases, emerges. It is shown for example, that when the ellipse’s eccentricity is set to zero to prescribe the circular paraboloid, the corresponding expression for the radiation agrees with those available in the open literature [1, 2].

## 1. INTRODUCTION

One of the earliest analytical treatments of the paraboloidal reflector was reported by Jones [3], whose analysis of the paraboloid reflector excited by short electric dipole, utilized the induced surface current approach to determine the reflector’s aperture distribution. Thereafter, quite a few other analytical investigations have been reported over the years, and some of the more noteworthy of these, as noted by Lorenzo et al. [9], include the surface current (or aperture field) integration approach, GTD/GO [5] combination approach, the Uniform Asymptotic Theory, and the Physical Theory of Diffraction, to mention a few. In a relatively recent development, Lashab et al. [4] described a wavelet-based (basis and testing functions) moment-method solution of a physical optics formulation of the radiation field problem for a large reflector antenna. Ergül and Gürel [10] using a Magnetic-Field Integral Equation (MFIE) formulation, also described a moment-method technique, in which the MFIE is numerically discretized with the use of RWG curl-conforming expansion and testing functions defined on planar triangulations.

This paper presents an Electric Field Integral Equation (EFIE) formulation of the paraboloidal reflector antenna problem based on the ‘parametric approach’ described elsewhere [6, 7]. This formulation is particularly suitable for a moment-method solution, as the radiation fields of the antenna are easily determined, once an approximation to the current density is available.

## 2. PROBLEM FORMULATION

The coordinate system description of an elliptical parabola of revolution is given in Figure 1.

Primed quantities represent source point and unprimed for field point.  $\bar{r}'$  is the distance from the focus (origin) point to the surface of the reflector ( $S$ ) whilst  $\bar{r}$  represents the distance from focus to the field (observation) point,  $P$ . Consequently,  $\bar{R}$  is the distance between the surface of the reflector and the observation point. According to [6], the position vector  $\bar{r}'$ , can be written in parametric form as:

$$\bar{r}' = at \cos \varphi' \hat{x} + bt \sin \varphi' \hat{y} + ht^2 \hat{z} \quad (1)$$

where

$a$  = minor axis;

$b$  = major axis;

$h$  = height;

$t$  = parameter.

The parameters ‘ $a$ ’ and ‘ $b$ ’ specified above can be related through eccentricity of the geometrical shape, so that a single equation will suffice to describe both of them and considerably simplify the



The magnetic vector potential, assuming perfect electric conductor, for the antenna is given as

$$\bar{A}(\theta, \varphi) = \frac{\mu t^2}{4\pi} \iint_s \bar{J}(\theta', \varphi') \frac{e^{-jkR}}{R} r^2(\theta') g(\theta') \sqrt{\sec^2 \beta + (4/\pi^2) k^2(\varphi') t^4 \tan^2 \beta} d\theta' d\varphi' \quad (6)$$

For a moment method solution of the foregoing problem, (6) can be re-written as

$$\begin{aligned} \bar{A}(\theta, \varphi) = & \frac{\mu}{4\pi} \iint_s J(\theta', \varphi') \frac{e^{-jkR}}{R} r^2(\theta') g(\theta') \left\{ (\sin \varphi' \tan \beta - (2/\pi) k(\varphi') \cos \varphi' t^2 \tan \beta) \hat{x} \right. \\ & \left. + (\cos \varphi' \tan \beta + (2/\pi) k(\varphi') \sin \varphi' t^2 \tan \beta) \hat{y} + \hat{z} \right\} d\theta' d\varphi' \end{aligned} \quad (7a)$$

where,

$$g(\theta') = \left[ \frac{\sin \theta'}{1 + \cos \theta'} + \frac{e^2 \sin 2\theta'}{(1 + \cos \theta')(1 - e^2 \cos^2 \theta')} \right] d\theta' \quad (7b)$$

To derive the expressions for far field region, the phase term reduces to

$$R = |\bar{r} - \bar{r}'| = \bar{r} - \bar{r}' \cdot \hat{a}_r \quad (8a)$$

and the amplitude term

$$R = r \quad (8b)$$

subsequently,

$$\bar{r}' \cdot \hat{a}_r = r(\theta') t \sin \theta [\cos(\varphi - \varphi') + (2/\pi) k(\varphi') \cot \theta t^2 \tan \beta] \quad (8c)$$

let

$$\zeta = \cos(\varphi - \varphi') + (2/\pi) k(\varphi') \cot \theta t^2 \tan \beta \quad (8d)$$

(8c) can be re-written as

$$\bar{r}' \bullet \hat{a}_r = r(\theta') t \sin \theta \zeta \quad (8e)$$

Substituting (8a), (8b), & (8e) into (7a), the equation becomes

$$\begin{aligned} \bar{A}(\theta, \varphi) = & \frac{\mu e^{-jkr}}{4\pi r} \int_0^{2\pi} \int_0^\pi \left\{ J(\theta', \varphi') e^{jkr(\theta') t \sin \theta \zeta} r^2(\theta') g(\theta') \right. \\ & (\sin \varphi' \tan \beta - (2/\pi) k(\varphi') \cos \varphi' t^2 \tan \beta) \hat{x} \\ & \left. + (\cos \varphi' \tan \beta + (2/\pi) k(\varphi') \sin \varphi' t^2 \tan \beta) \hat{y} + \hat{z} \right\} d\theta' d\varphi' \end{aligned} \quad (9)$$

The generalized formulated magnetic vector potential is presented in Equation (9).

For circular paraboloid, eccentricity will be equal to zero and consequently, Equation (9) reduces to

$$\begin{aligned} \bar{A}(\theta, \varphi) = & \frac{\mu e^{-jkr}}{4\pi r} \int_0^{2\pi} \int_0^\pi \left\{ J(\theta', \varphi') e^{jkr(\theta') t \sin \theta \xi} \rho'^2 \sec\left(\frac{\theta'}{2}\right) \sin \theta' \right. \\ & (\sin \varphi' \tan \beta - (2/\pi) k(\varphi') \cos \varphi' t^2 \tan \beta) \hat{x} \\ & \left. + (\cos \varphi' \tan \beta + (2/\pi) k(\varphi') \sin \varphi' t^2 \tan \beta) \hat{y} + \hat{z} \right\} d\theta' d\varphi' \end{aligned} \quad (10a)$$

$$\xi = \cos(\varphi - \varphi') + \cot \theta t^2 \tan \beta \quad (10b)$$

Equation (9) is in agreement with similar expressions reported in [2]. Equation (10a) can be re-written in terms of Cartesian coordinates as

$$\begin{aligned}
 \bar{A}_x(\theta, \varphi) &= \frac{\mu e^{-jkr} \tan \beta}{4\pi r} \int_0^{2\pi} \int_0^\pi J(\theta', \varphi') e^{jkr(\theta')t \sin \theta \xi} \rho'^2 \sec\left(\frac{\theta'}{2}\right) \sin \theta' \\
 &\quad (\sin \varphi' - (2/\pi) k(\varphi') t^2 \cos \varphi') d\theta' d\varphi' \\
 \bar{A}_y(\theta, \varphi) &= \frac{\mu e^{-jkr} \tan \beta}{4\pi r} \int_0^{2\pi} \int_0^\pi J(\theta', \varphi') e^{jkr(\theta')t \sin \theta \xi} \rho'^2 \sec\left(\frac{\theta'}{2}\right) \sin \theta' \\
 &\quad (\cos \varphi' + (2/\pi) k(\varphi') t^2 \sin \varphi') d\theta' d\varphi' \\
 \bar{A}_z(\theta, \varphi) &= \frac{\mu e^{-jkr}}{4\pi r} \int_0^{2\pi} \int_0^\pi \{ J(\theta', \varphi') e^{jkr(\theta')t \sin \theta \xi} \rho'^2 \sec\left(\frac{\theta'}{2}\right) \sin \theta' d\theta' d\varphi'
 \end{aligned} \tag{11}$$

So that using the identities given as

$$\begin{aligned}
 A_\theta &= \cos \theta \cos \varphi A_x + \cos \theta \sin \varphi A_y + \sin \theta A_z \\
 A_\varphi &= -\sin \varphi A_x + \cos \varphi A_y
 \end{aligned}$$

we find that the vector magnetic potentials, in both the theta and phi direction, emerge as:

$$\begin{aligned}
 \bar{A}_\theta &= \frac{\mu e^{-jkr} \cos \theta \tan \beta}{4\pi r} \int_0^{2\pi} \int_0^\pi J(\theta', \varphi') e^{jkr(\theta')t \sin \theta \xi} \rho'^2 \sec\left(\frac{\theta'}{2}\right) \sin \theta' \\
 &\quad \cdot [\sin(\varphi - \varphi') - t^2 \cos(\varphi - \varphi') - \tan \theta \cot \beta] d\theta' d\varphi'
 \end{aligned} \tag{12a}$$

$$\begin{aligned}
 \bar{A}_\varphi &= \frac{\mu e^{-jkr} \tan \beta}{4\pi r} \int_0^{2\pi} \int_0^\pi J(\theta', \varphi') e^{jkr(\theta')t \sin \theta \xi} \rho'^2 \sec\left(\frac{\theta'}{2}\right) \\
 &\quad \sin \theta' [t^2 \sin(\varphi - \varphi') - \cos(\varphi - \varphi')] d\theta' d\varphi'
 \end{aligned} \tag{12b}$$

At far field region,

$$\bar{E}_\theta = -j\omega \bar{A}_\theta \tag{13a}$$

and

$$\bar{E}_\varphi = -j\omega \bar{A}_\varphi \tag{13b}$$

substituting (12a) and (12b) into (13a) and (13b) respectively yields

$$\begin{aligned}
 \bar{E}_\theta &= -\frac{j\omega \mu e^{-jkr} \cos \theta \tan \beta}{4\pi r} \int_0^{2\pi} \int_0^\pi J(\theta', \varphi') e^{jkr(\theta')t \sin \theta \xi} \rho'^2 \sec\left(\frac{\theta'}{2}\right) \sin \theta' \\
 &\quad \cdot [\sin(\varphi - \varphi') - t^2 \cos(\varphi - \varphi') - \tan \theta \cot \beta] d\theta' d\varphi'
 \end{aligned} \tag{14a}$$

$$\begin{aligned}
 \bar{E}_\varphi &= -\frac{j\omega \mu e^{-jkr} \tan \beta}{4\pi r} \int_0^{2\pi} \int_0^\pi J(\theta', \varphi') e^{jkr(\theta')t \sin \theta \xi} \rho'^2 \sec\left(\frac{\theta'}{2}\right) \\
 &\quad \sin \theta' [t^2 \sin(\varphi - \varphi') - \cos(\varphi - \varphi')] d\theta' d\varphi'
 \end{aligned} \tag{14b}$$

### 3. DISCUSSION OF RESULTS

It is apparent from the formulation that the use of Equation (2a) for the major and minor axes makes it easy to specialize to circular type. Equation (6) is a general Electromagnetic formulation that can be solved using numerical methods. In particular, a method of moment solution will provide numerical values for the unknown current density to facilitate the evaluation of the antenna's radiation fields with the use of Equations (14a) and (14b).

#### 4. CONCLUSION

A parametric formulation of the paraboloid reflector problem, starting with the elliptical geometry, has been developed in this presentation. A check on the validity of the analytical results obtained was provided by a specialization to the circular type. Integral expressions developed in the paper, for the antenna's radiation fields are particularly suitable for a moment method solution, possibly using wavelet basis and testing functions of the type described in [4].

#### REFERENCES

1. Salman, O. I., A. S. Tahir, and A. A. Al-Hilo, "Theoretical analysis of the radiation fields of parabolic reflector antenna fed by circular shaped microstrip antennas," *Basrah Journal of Science (A)*, Vol. 30, No. 2, 30–43, 2012.
2. Maclean, T. S. M., *Principles of Antennas*, Cambridge University Press, New York, 1986.
3. Jones, E. M. T., "Paraboloid reflector and hyperboloid lens antennas," *IRE Trans. Antennas Propagat.*, Vol. 701, No. 2, 119–127, Jul. 1954.
4. Lashab, M., C. Zebiri, and F. Benabdelaziz, "Wavelet-based moment method and physical optics use on large reflector antennas," *Progress In Electromagnetics Research M*, Vol. 2, 189–200, 2008.
5. Merabtine, N., A. Boualleg, and M. Benslama, "Analysis of radiation patterns and feed illumination of the reflector antenna using the physical and geometrical optics," *Semiconductor Physics, Quantum Electronics & Optoelectronics*, Vol. 9, No. 2, 53–57, 2006.
6. Tuma, J. J., *Engineering Mathematics Handbook*, 3rd Edition, McGraw-Hill Book Company, 2006.
7. Adekola, S. A., A. Ike Mowete, and H. A. Muhammed, "A matrix-vector-potential analysis of the bi-elliptical toroidal helical antenna," *PIERS Proceedings*, 825–829, Marrakesh, Morocco, Mar. 20–23, 2011.
8. Adekola, S. A., A. I. Mowete, and A. A. Ayorinde, "Concerning a generalized formulation for the EM fields radiated by thin-wire antennas," *Proc. of Nat. Conf.*, Dept. Elec. & Elec. Eng., Univ. of Lagos, Jul. 15–17, 2006.
9. Martinez Lorenzo, J. A., A. G. Pino, I. Vega, M. Arias, and O. Rubilinos, "ICARA: Induced-current analysis of reflector antennas," *IEEE Antennas and Propagat. Magazine*, Vol. 47, No. 2, 82–100, Apr. 2005.
10. Ergül, Z. and O. Gürel, "The use of curl-conforming basis function for the magnetic-field integral equation," *IRE Trans. Antennas Propagat.*, Vol. 54, No. 7, 1917–1925, Jul. 2006.