A Uniform Asymptotic Solution for Diffraction by a Right-angled Dielectric Wedge

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Abstract — The aim of this work is to propose an approximate asymptotic solution for the field diffracted by a lossless right-angled dielectric wedge in the case of a plane wave having normal incidence with respect to the edge. The diffraction problem is tackled by considering two observation regions: the inner region of the wedge and the surrounding free-space. For each of them, the electric and magnetic surface currents involved in the radiation integrals are determined in the Physical Optics approximation. Useful analytical manipulations and uniform asymptotic evaluations allow one to obtain the diffracted field in terms of the Fresnel’s reflection and transmission coefficients of the structure and the transition function of the Uniform Geometrical Theory of Diffraction. The proposed solution for the diffracted field compensates the Geometrical Optics field discontinuities and its accuracy is well-assessed by using Finite Difference Time Domain results.

1. INTRODUCTION
Diffraction by a wedge is a well-covered research topic due to its relevance for practical applications, but the results available in the scientific literature mainly concern impenetrable structures. This is due to difficulties encountered by researchers in the case of penetrable materials, where a complex coupling between the external and the internal region arises. The existing approaches can be characterized by either trying to provide analytical or heuristic approximate solutions, or trying to solve the problem in an exact sense using combined analytical-numerical techniques. With reference to the diffraction by a dielectric wedge, significant but not conclusive contributions can be found in [1–4].

A Uniform Asymptotic Physical Optics (UAPO) solution for predicting the field diffracted by a lossless right-angled dielectric wedge when illuminated by a uniform plane wave at normal incidence (see Fig. 1) is proposed in this paper. The considered problem is splitted into two subproblems concerning the regions external and internal to the wedge. With reference to the outer problem, equivalent electric and magnetic PO surface currents lying on the external faces of the wedge are assumed as sources in the standard radiation integral. Useful analytical manipulations and uniform asymptotic evaluations of the resulting integrals give the field diffracted in the space surrounding the wedge in terms of the Fresnel’s reflection coefficients of the structure and the transition function of the Uniform Geometrical Theory of Diffraction (UTD) [5]. The inner problem is solved by determining equivalent electric and magnetic PO surface currents on the internal faces of the wedge. They are related to the field transmitted into the structure. Once such currents are known, the diffracted field is evaluated by an approach like that used for the outer problem. Numerical results show that the here derived solution for the diffracted field compensates the discontinuities of the Geometrical Optics (GO) field at the shadow boundaries in the external and

Figure 1: Geometry of the problem.

Figure 2: Shadow boundaries.
internal regions. Moreover, it results to be very accurate as confirmed by comparisons between the UAPO-based approach results and those obtained via the reliable Finite Difference Time Domain (FDTD) technique.

2. DIFFRACTED FIELD: UAPO SOLUTION

Let us consider the problem of plane-wave diffraction by the edge of a lossless right-angled dielectric wedge illuminated at normal incidence (see Fig. 1). The surface $S_0$ is located at $\phi = 0$ whereas $S_{\pi/2}$ corresponds to $\phi = 3\pi/2$. They divide the space into the outer region ($0 < \phi < 3\pi/2$) and the inner region ($3\pi/2 < \phi < 2\pi$). The incident plane wave is linearly polarized along the z-axis and propagates in the direction fixed by $\phi'$ so as both surfaces result to be illuminated ($\pi/2 < \phi' < \pi$).

Accordingly, there are two reflection boundaries in the region surrounding the wedge and two transmission boundaries in the internal region (see Fig. 2).

It is convenient to split the original problem in accordance to the equivalence theorem, and to consider two sub-problems relevant to the regions external and internal to the wedge.

2.1. Outer Region

The electric field scattered by the wedge can be represented by the well-known radiation integral:

$$
E_{\text{out}}^s = -j k_0 \int_S \left[ \left( \mathbb{I} - \tilde{R} \mathbb{R} \right) (\zeta_0 \mathbb{J}_s) + \mathbb{J}_{\text{ms}} \times \tilde{R} \right] G(r, r') \, dS
$$

wherein $S = S_{0,\text{out}} \cup S_{\pi/2,\text{out}}$, $\mathbb{J}_s$ and $\mathbb{J}_{\text{ms}}$ are the equivalent electric and magnetic surface currents over $S$, $G(r, r')$ is the Green’s function, $\zeta_0$ and $k_0$ are the impedance and propagation constant for the free-space, $r$ and $r'$ denote the observation and source points, respectively, $\mathbb{R}$ is the unit vector from the radiating element at $r'$ to the observation point, and $\mathbb{I}$ is the $(3 \times 3)$ identity matrix. Due to the linearity of the operator in (1), $S_{0,\text{out}}$ and $S_{\pi/2,\text{out}}$ contribute separately to the scattered field, so that $E_{\text{out}}^s = E_{0,\text{out}}^s + E_{\pi/2,\text{out}}^s$. The key steps of the proposed approach are presented only with reference to the diffracted field $E_{\pi/2,\text{out}}^d$ related to $E_{0,\text{out}}^s$. For this contribution, the corresponding PO surface currents can be so expressed:

$$
\zeta_0 \mathbb{J}_s^{\text{out}} = E_0 (1 - R_0) \sin \phi' e^{jk_0 x \cos \phi' \hat{z}}
$$

$$
\mathbb{J}_{\text{ms}}^{\text{out}} = -E_0 (1 + R_0) e^{jk_0 x \cos \phi' \hat{x}}
$$

in which $E_0$ denotes the incident field at the origin and $R_0$ is the Fresnel’s reflection coefficient associated to $S_0$. Since diffraction is confined to the Keller’s cone, the approximation $\tilde{R} \approx \hat{s}$ ($\hat{s}$ is the diffraction direction) is permitted for evaluating the edge diffracted field. As a consequence, after analytical manipulations, the radiation integral associated to $S_{0,\text{out}}$ can be written as follows:

$$
E_{0,\text{out}}^s \approx \hat{s} E_0 \left[ (1 - R_0) \sin \phi' - (1 + R_0) \sin \phi \right] \frac{1}{4\pi j} \int_C e^{-jk_0 \rho \cos (\alpha + \phi)} \cos \alpha + \cos \phi' \, d\alpha
$$

The integration path $C$ in (4) is shown in Fig. 3. The application of the Steepest Descent Method allows one to write the integral in (4) as a typical diffraction integral, and a uniform asymptotic evaluation of this last gives the diffraction contribution:

$$
E_{0,\text{out}}^d = \hat{s} E_0 \left[ (1 - R_0) \sin \phi' - (1 + R_0) \sin \phi \right] e^{-jk_0 \rho \cos (\phi + \phi')} \left[ \frac{2k_0 \rho \cos^2 \left( \frac{\phi + \phi'}{2} \right)}{2\sqrt{2\pi k_0}} \right] F_1 \left[ \frac{2k_0 \rho \cos^2 \left( \frac{\phi + \phi'}{2} \right)}{\cos \phi + \cos \phi'} \right] e^{-jk_0 \rho \cos \phi} \frac{1}{\sqrt{\rho}}
$$

where $F_1(\cdot)$ is the UTD transition function [5], and $+ (-)$ sign applies when $0 < \phi < \pi$ ($\pi < \phi < 3\pi/2$).

With reference to the diffracted field $E_{\pi/2,\text{out}}^d$ related to $E_{\pi/2,\text{out}}^s$, the corresponding PO surface currents

$$
\zeta_0 \mathbb{J}_s^{\text{out}} = -E_0 \left( 1 - R_{\pi/2} \right) \cos \phi' e^{jk_0 y \sin \phi' \hat{z}}
$$

$$
\mathbb{J}_{\text{ms}}^{\text{out}} = -E_0 \left( 1 + R_{\pi/2} \right) e^{jk_0 y \sin \phi' \hat{y}}
$$

provide

\[ E_{\pi/2,\text{out}}^d = \hat{z} E_0 \left[ (1 + R_{\pi/2}) \cos \phi - (1 - R_{\pi/2}) \cos \phi' \right] \frac{e^{-j\pi/4}}{2\sqrt{2\pi k_0}} \]
\[ \times \frac{F_l}{\cos (3\pi/2 - \phi) + \cos (3\pi/2 - \phi')} \frac{2\sqrt{2\pi k_d}}{\sqrt{\rho}} \]
\[ \cdot \cos (\pi - \phi) + \cos (\pi - \phi') e^{-jk_d \rho \sqrt{\rho}} \]
\[ \left( \cos \phi + \cos \theta_t^l \right) \frac{e^{-j\pi/4}}{2\sqrt{2\pi k_d}} \frac{2k_d \rho \cos^2 \left( \frac{\pi/2 - \phi + \cos^{-1} \left( \cos \phi'/\sqrt{\varepsilon_r} \right) \rho}{2 \sqrt{\varepsilon_r}} \right)}{\sin \phi' + \sqrt{\varepsilon_r} \sin \phi} \]
\[ \left( \cos \phi - \cos \theta_t^l \right) \frac{e^{-j\pi/4}}{2\sqrt{2\pi k_d}} \frac{2k_d \rho \cos^2 \left( \frac{\pi/2 - \phi - \cos^{-1} \left( \cos \phi'/\sqrt{\varepsilon_r} \right) \rho}{2 \sqrt{\varepsilon_r}} \right)}{\sin \phi' + \sqrt{\varepsilon_r} \sin \phi} \]

where + (−) sign applies when \( \pi/2 < \phi < 3\pi/2 \) \( (0 < \phi < \pi/2) \).

### 2.2. Inner Region

The inner problem is solved by using an approach like that adopted for the outer problem. Accordingly, the resulting UAPO field diffracted in the internal region of the wedge is given by

\[ E_{\text{in}}^d = E_{0,\text{in}}^d + E_{\pi/2,\text{in}}^d, \]

where

\[ E_{0,\text{in}}^d = \hat{z} E_0 T_0 \sqrt{\varepsilon_r} \left[ \sin \phi - \cos \theta_t^l \right] \frac{e^{-j\pi/4}}{2\sqrt{2\pi k_d}} \frac{F_l}{\cos \phi' + \sqrt{\varepsilon_r} \cos \phi} \]
\[ \cdot \left( \cos \phi + \cos \theta_t^l \right) \frac{e^{-j\pi/4}}{2\sqrt{2\pi k_d}} \frac{2k_d \rho \cos^2 \left( \frac{\pi/2 - \phi + \cos^{-1} \left( \cos \phi'/\sqrt{\varepsilon_r} \right) \rho}{2 \sqrt{\varepsilon_r}} \right)}{\sin \phi' + \sqrt{\varepsilon_r} \sin \phi} \]
\[ \left( \cos \phi - \cos \theta_t^l \right) \frac{e^{-j\pi/4}}{2\sqrt{2\pi k_d}} \frac{2k_d \rho \cos^2 \left( \frac{\pi/2 - \phi - \cos^{-1} \left( \cos \phi'/\sqrt{\varepsilon_r} \right) \rho}{2 \sqrt{\varepsilon_r}} \right)}{\sin \phi' + \sqrt{\varepsilon_r} \sin \phi} \]

In (9) and (10), \( T_0 \) \( (T_{\pi/2}) \) and \( \theta_t^l \) \( (\theta_{\pi/2}^l) \) are the transmission coefficient and the transmission angle (see Fig. 2) related to \( S_0 \) \( (S_{\pi/2}) \), and \( k_d = k_0 \sqrt{\varepsilon_r} \) is the propagation constant in the wedge region by assuming a non-magnetic dielectric having relative permittivity \( \varepsilon_r \).

### 3. NUMERICAL RESULTS

Numerical results are provided to assess the correctness and effectiveness of the proposed solution. They are relevant to a wedge characterized by \( \varepsilon_r = 4 \) and a plane wave propagating in the direction \( \phi' = 135^\circ \). The field is evaluated over a circular path with radius \( \rho = 8 \lambda_0 \), \( \lambda_0 \) being the free-space wavelength. Fig. 4 shows the GO field and the proposed UAPO diffracted field. As expected, the GO field has four discontinuities at the shadow boundaries \( R_{B_0} \) \( (\phi = 45^\circ) \), \( R_{B_{\pi/2}} \) \( (\phi = 225^\circ) \), \( T_{B_0} \) \( (\phi = 291^\circ) \) and \( T_{B_{\pi/2}} \) \( (\phi = 339^\circ) \). On the other hand, the diffracted field presents peaks at such boundaries. It ensures the continuity of the total field into the wedge and in the surrounding space (see Fig. 5). The accuracy of the here derived solution has been proved by means of comparisons with the results obtained via an “ad hoc” developed FDTD code. The comparison relevant to the considered case is reported in Fig. 6 and confirms this statement.
4. CONCLUSIONS

Uniform asymptotic solutions for evaluating the field diffracted by a right-angled dielectric wedge in the inner and outer regions have been presented in this paper. They are expressed in terms of the standard Fresnel’s coefficients and the UTD transition function, are easy to handle, compensate the GO field discontinuities at the shadow boundaries and give accurate results.

REFERENCES