Observation of Geometric Resonance in a Corrugated Waveguide

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Abstract— Geometric resonance phenomenon in a corrugated waveguide is investigated both theoretically and experimentally. By solving wave equation under the periodic boundary condition in a corrugated waveguide, geometric resonance conditions are derived. This resonance is associated with the waveguide thickness and the phase shift between the corrugated walls of the waveguide. A corrugated waveguide is fabricated and measured by an HP 8510c Vector Network Analyzer, measurement results agree with the theoretical prediction. This work is meaningful for microwave communications, quantum physics, photonic crystals, integrated optical and electronic devices. The theory is also of significance to the research on periodic quantum well.

1. INTRODUCTION
Wave propagation in periodic structure is a classical problem. It is now still of great interest due to its applications in microwave communications, quantum physics, photonic crystals, integrated optical and electronic devices [1–7]. Although this topic has been investigated for decades, there are still some issues, such as electromagnetic wave propagation in a corrugated waveguide, need further study.

Wave propagation in periodic structure can be characterized by Bragg’s Law. Under certain conditions, Bragg reflection, which will lead to a forbidden gap in transmission spectrum, can be observed. In this paper, resonance due to geometry configurations is presented. For a carefully chosen waveguide thickness and phase shift between corrugated walls, geometric resonance will occur, in this case, a band gap will open around one of the higher mode cutoff frequencies; meanwhile, the corrugation of the waveguide walls will lead to a shift for lower mode cutoff frequency.

2. WAVE EQUATION AND BOUNDARY CONDITIONS
We assume electromagnetic (EM) wave travels along x-direction in Cartesian coordinates. Waveguide with periodically corrugated walls is considered. The corrugated walls have the profiles of $y_d = \frac{d}{2} + \xi \cos \left( \frac{2\pi}{a} x \right)$, $y_{-d} = -\frac{d}{2} + \rho \cos \left( \frac{2\pi}{a} x + \theta \right)$ for upper and lower walls respectively, where $d$ is the waveguide thickness, $a$ is the period of corrugation, $\xi$, $\rho$ and $\theta$ are amplitudes and phase shift of the corrugated walls. The configuration is shown in Figure 1.

If we denote $\varphi(x, y, z)$ as the $z$-component of the $E$ field of the TE wave in the waveguide, wave propagation in this corrugated waveguide can be characterized using the two-dimensional wave equation:

$$\nabla^2 \varphi + \frac{\omega^2}{c^2} \varphi = 0$$

subject to the following boundary conditions:

$$\varphi(x, y_d) = \varphi(x, y_{-d}) = 0$$

![Figure 1: Corrugated wall configurations.](image-url)
where \( \omega \) is the frequency of the EM wave in \( \text{rad/s} \), \( c \) is the propagating velocity of the wave in \( \text{m/s} \). Due to the periodicity of the waveguide boundary, Eq. (2) can be expanded into the following Fourier series:

\[
\varphi(x, y) = \sum_n [a_n \cos(k_{yn}y) + b_n \sin(k_{yn}y)] \times e^{j(k_n + n\frac{2\pi}{d})x}
\]

where \( a_n \) and \( b_n \) are Fourier series coefficients, \( k_{yn} \) and \( k_x \) are components of wave vector along \( y \)-and \( x \)-directions.

For small corrugation, i.e., \( \xi \ll 1 \), and \( \frac{d}{\lambda} \ll 1 \), it is sufficient to consider the first three harmonics [7]. Based on Eq. (2) and Eq. (3), the following characterization equation is used to find out allowable \( k_{yn} \):

\[
\tan(k_{0d}) = \frac{1}{2} \times (\xi^2 + \rho^2) \times \frac{k_0k_1}{\tan(k_1d)} - \frac{\rho\xi k_0k_1 \cos \theta}{\cos(k_0d) \sin(k_1d)}
\]

where \( k_0^2 = k_1^2 = k_0^2 + q^2 = k_0^2 + (\frac{2\pi}{\lambda})^2 \) is the wave number of the first harmonic. In Eq. (4) and the following sections, we will drop the footnote \( n \).

### 3. CUTOFF FREQUENCY SHIFT AND GEOMETRIC RESONANCE

The solution to Eq. (4) can be obtained by the method of successive approximation. For smooth waveguide, \( \rho = \xi = 0 \), the cutoff frequency of \( \text{TE}_{0p} \) mode is:

\[
f_{0p} = \frac{c}{\sqrt{\varepsilon}} \times \frac{p\pi}{d}, \quad p = 1, 2, 3 \ldots \text{ is the order of TE wave}
\]

For corrugated waveguide, \( \rho \neq 0 \) and \( \xi \neq 0 \), we have the cutoff frequency:

\[
f'_{0p} = f_{0p} \left[ 1 + \frac{1}{2} \times (\xi^2 + \rho^2) \times \frac{k_1}{\tan(k_1d)} + (-1)^{p+1} \frac{\rho\xi k_1 \cos \theta}{2d \times \sin(k_1d)} \right]
\]

Comparing with a smooth waveguide with the same average thickness, the cutoff frequency shift is:

\[
\delta f'_{0p} = \frac{1}{2} \times (\xi^2 + \rho^2) \times \frac{k_1}{\tan(k_1d)} + (-1)^{p+1} \frac{\rho\xi k_1 \cos \theta}{2d \times \sin(k_1d)}
\]

From Eq. (4) and Eq. (6) the resonance condition is:

\[
k_{0p}^2 = k_{1m}^2 + \left( \frac{2\pi}{d} \right)^2
\]

where \( k_{1m} = k_1 = \frac{m\pi}{d}, m = 1, 2, 3 \ldots \) is the order of resonance.

When geometric resonance occurs, one can observe the opening of a forbidden gap in the transmission spectrum as well as cutoff frequency shift.

1) Asymmetric waveguide: In this case, there is no phase difference between corrugated walls, \( \theta = 0 \). If the TE wave mode \( p \) and harmonic order \( m \) are both even or both odd, the resonance frequency is:

\[
f_{pm}^\pm = f_{0p} \left( 1 \pm \frac{1}{\sqrt{2}} \times \frac{m}{p} \times \frac{|\rho - \xi|}{d} \right)
\]

where \( f_{pm}^+ \) and \( f_{pm}^- \) corresponds to upper and lower cutoff frequencies of the band gap; the gap width could be determined by:

\[
\delta f_{pm} = \sqrt{2} \times \frac{m}{p} \times \frac{|\rho - \xi|}{d} \times f_{0p}
\]

If the \( p \) is odd and \( m \) is even or on contrary,

\[
f_{pm}^\pm = f_{0p} \left( 1 \pm \frac{1}{\sqrt{2}} \times \frac{m}{p} \times \frac{|\rho + \xi|}{d} \right)
\]

\[
\delta f_{pm} = \sqrt{2} \times \frac{m}{p} \times \frac{|\rho + \xi|}{d} \times f_{0p}
\]

2) Symmetric waveguide: In this case, phase difference between corrugated walls \( \theta = \pi \). If the TE wave mode \( p \) and harmonic order \( m \) are both even or both odd, the resonance frequency is determined by Eq. (11), bad gap width is calculated by Eq. (12); If the \( p \) is odd and \( m \) is even or on contrary, resonance frequency can be found by Eq. (9), band gap is determined by Eq. (10).
4. EXPERIMENTAL OBSERVATION OF GEOMETRIC RESONANCE

A waveguide with corrugated lower and upper walls is fabricated. The lower wall is fixed, while the upper one is free to move, which makes the waveguide thickness and phase shifts between corrugated walls can be adjusted as desired. Measurements are done on a HP 8510C Vector Network Analyzer, proper antenna pairs are used. Both corrugated walls have periodic of 3.15 cm, amplitudes for upper and lower corrugation are the same, $\rho = \xi = 0.415$ cm. Since $\frac{\rho}{d} = \frac{\xi}{d} = 0.15 \ll 1$, Eq. (4) holds. According to Eq. (8), the average thickness of the waveguide is chosen to be $d = 2.73$ cm, we have $p = 2$ and $m = 1$, geometric resonance occurs.

We will begin our measurement with observing the cutoff frequency shift of TE$_{01}$ mode. It can be seen from Figure 2(a) that the smooth waveguide has a cutoff frequency of 5.53 GHz which is very close to the theoretical value of 5.5 GHz. According to Eq. (6) and Eq. (7), the cutoff frequency of TE$_{01}$ mode should shift to 6.02 GHz. The measured result in Figure 2(b) shows the cutoff frequency shifted to 5.87 GHz, which generally agrees with our prediction. The difference might be caused of unevenness of the corrugation.

It is supposed that we should have a band gap around the cutoff frequency of TE$_{20}$ mode for the corrugated waveguide. According to Eq. (8), we have $p = 2$, $m = 1$. For symmetric waveguide, $\theta = \pi$, since $p$ is even, $m$ is odd, substitute data to Eq. (9), it is supposed that the gap is minimized in this case; for asymmetric waveguide, $\theta = 0$, using Eq. (11) and Eq. (12), there should have a gap between 9.82 GHz to 12.18 GHz.

It is seen from Figure 3 that a band gap opens from 9.875 GHz to 12.08 GHz, this measurement result agrees with our theoretical prediction. When the phase shift between corrugated walls is

![Figure 2: TE$_{01}$ mode cutoff frequency shift. (a) TE$_{01}$ mode cutoff frequency of smooth waveguide. (b) TE$_{01}$ mode cutoff frequency of corrugated waveguide.](image)

![Figure 3: Band gap due to geometric resonance.](image)
\( \theta = \pi \), the forbidden gap is minimized, and it is very close to the performance of a smooth waveguide which has the same average thickness.

5. CONCLUSIONS

Resonance caused by specific geometric configurations is investigated both theoretically and experimentally. The geometric resonance will alter transmission spectrum of a corrugated waveguide. By solving wave equation under periodic boundary conditions, geometric resonance conditions are derived; cutoff frequency shift and opening of a bag gap are predicted. Waveguide with periodically corrugated walls is fabricated, under carefully chosen configurations, geometric resonance occurs in the waveguide, band gap in the transmission spectrum as well as cutoff frequency shift are observed. Measurement results show good agreement with theoretical calculations. Unevenness of the periodicity of the corrugations may lead to differences between theoretical prediction and measurement results.

REFERENCES