Utilizing the Radiative Transfer Equation in Optical Tomography

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Abstract— We propose a method which utilizes the radiative transfer equation in optical tomography. In this approach, the radiative transfer equation is used as light propagation model in those regions in which the assumptions of the diffusion theory are not valid and the diffusion approximation is used elsewhere. Both the radiative transfer equation and the diffusion approximation are numerically solved with a finite element method. In the finite element solution of the radiative transfer equation, both the spatial and angular discretizations are implemented in piecewise linear bases.

1. INTRODUCTION

In diffuse optical tomography (DOT), images of optical properties of tissues are derived based on measurements of near-infrared light on the surface of the object. The image reconstruction in DOT is a non-linear ill-posed inverse problem. Thus, even small errors in the measurements or modelling can cause large errors in the reconstructions. Therefore, computationally feasible forward models that describe light propagation within the medium accurately are needed.

The radiative transfer equation (RTE) is widely accepted as an accurate model for light propagation in tissues \cite{2}. However, it is computationally expensive, and therefore the most most typical approach in DOT has been to use the diffusion approximation (DA) to the RTE as the forward model. The DA is basically a special case of the first order spherical harmonics approximation to the RTE, and thus it has some limitations. Firstly, the medium must be scattering dominated, and secondly, light propagation cannot be modelled accurately close to the collimated light sources and boundaries \cite{2}.

To overcome the limitations of the diffusion theory, different hybrid methods that combine diffusion approximation with other models have been developed. The radiosity–diffusion model \cite{3} can be used for turbid medium with non-scattering regions. However, it does not model light propagation accurately in low-scattering regions or close to the collimated light sources. Methods which combine Monte Carlo simulation with diffusion theory have also been reported \cite{8, 18}. Monte Carlo is known to describe light propagation accurately. However, it has the disadvantage of requiring a long computation time. Moreover, the hybrid Monte Carlo-diffusion methods often require iterative mapping between the models which increases the computation time even more.

Methods in which the RTE is utilized in DOT by combining it with diffusion theory have been developed. These include the coupled transport and diffusion model \cite{6} and the hybrid radiative transfer-diffusion approach \cite{15} where the transport and diffusion models are coupled and iteratively solved. A coupled RTE–DA model was proposed by the authors in \cite{16}. In the coupled RTE–DA model, the RTE is used as the forward model in sub-domains in which the assumptions of the DA are not valid and the DA is used elsewhere in the domain. The RTE and the DA are coupled through boundary conditions between the sub-domains and they are solved simultaneously using a finite element method (FEM). In this paper, the coupled RTE–DA model is utilized in light transport simulations in realistic two-dimensional (2D) head geometry from a new-born infant’s head.

2. METHODS

2.1. Radiative Transfer Equation

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or $3$ denote the physical domain and $\partial \Omega$ the boundary of the domain, and let $\hat{s} \in S^{n-1}$ denote a unit vector in the direction of interest. A widely accepted model for light
propagation in tissues is the radiative transfer equation. The frequency domain version of the RTE is of the form

\[
\left( \frac{i \omega}{c} + \hat{s} \cdot \nabla + \mu_s + \mu_a \right) \phi(r, \hat{s}) = \mu_s \int_{S^{n-1}} \phi(r, \hat{s}') \Theta(\hat{s} \cdot \hat{s}') d\hat{s}' + q(r, \hat{s})
\]

(1)

where \(i\) is the imaginary unit, \(\omega\) is the angular modulation frequency of the input signal, \(c\) is the speed of light in the medium, \(\mu_s\) and \(\mu_a\) are the scattering and absorption coefficients of the medium, respectively, \(\phi(r, \hat{s})\) is the radiance, and \(q(r, \hat{s})\) is the source inside \(\Omega\) [2]. The kernel \(\Theta(\hat{s} \cdot \hat{s}')\) is the scattering phase function which describes the probability that a photon with an initial direction \(\hat{s}'\) will have a direction \(\hat{s}\) after a scattering event. In DOT, we use a vacuum boundary condition for the RTE which assumes that no photons travel in an inward direction at the boundary \(\partial \Omega\).

Including a boundary source \(\phi_0\) at the source position \(\varepsilon_j \subset \partial \Omega\), the boundary condition can be written as

\[
\phi(r, \hat{s}) = \begin{cases} 
\phi_0(r, \hat{s}), & r \in \bigcup_j \varepsilon_j, \quad \hat{s} \cdot \hat{n} < 0, \\
0, & r \in \partial \Omega \setminus \bigcup_j \varepsilon_j, \quad \hat{s} \cdot \hat{n} < 0,
\end{cases}
\]

(2)

where \(\hat{n}\) is the outward unit normal on \(\partial \Omega\).

A finite element (FE) solution for the steady-state RTE was derived in [7]. Since then, both the forward and inverse problems have been studied, see e.g., [1, 5, 12]. In the frequency domain, numerical solutions of the RTE have been obtained using finite volume-discrete ordinates method [13], finite element method [15, 17], finite element-spherical harmonics method [19], and finite difference-discrete ordinates method [11]. In this work, the RTE is solved with the FEM. Both the spatial and angular discretizations are implemented in piecewise linear bases.

### 2.2. Diffusion Approximation

In DOT, light propagation in tissues is usually modelled with the diffusion approximation to the RTE. The frequency domain version of the DA is of the form

\[
-\nabla \cdot \kappa \nabla \Phi(r) + \mu_a \Phi(r) + \frac{i \omega}{c} \Phi(r) = q_0(r)
\]

(3)

where \(\kappa = (\mu_a + \mu_s')^{-1}\) is the diffusion coefficient where \(\mu_s' = (1 - g_1)\mu_s\) is the reduced scattering coefficient, and \(g_1\) is the mean of the cosine of the scattering angle, \(\Phi(r) = \int_{S^{n-1}} \phi(r, \hat{s}) d\hat{s}\) is the photon density, and \(q_0(r)\) is a source inside \(\Omega\). The DA can not satisfy the boundary condition (2). Instead it is often replaced by an approximation that the total inward directed photon current is zero. Further, by taking into account the mismatch between the refractive indices of the medium and surrounding medium, a Robin type boundary condition can be derived. It is of the form

\[
\Phi(r) + \frac{1}{2\gamma_n} \kappa A \frac{\partial \Phi(r)}{\partial \hat{n}} = 0, \quad r \in \partial \Omega
\]

(4)

where \(\gamma_n\) is a dimension-dependent constant which takes values \(\gamma_2 = 1/\pi\) and \(\gamma_3 = 1/4\) and \(A\) is a parameter governing the internal reflection at the boundary \(\partial \Omega\), with \(A = 1\) for the case of matched refractive index [2, 10]. The light sources at \(\partial \Omega\) are usually modelled in the DA either by the collimated source model or the diffuse boundary source model. In case of the collimated source model, the light source is modelled as an isotropic point source located at a depth \(1/\mu_s'\) below the source site. In case of the diffuse boundary source model, the source is modelled as a diffuse boundary current \(I_s\) at the source position.

The finite element approximation of the DA has been derived in several papers, see e.g., [4]. In this study, the FE-approximation of the DA is constructed similarly as in [15].

### 2.3. Coupled RTE–DA Model

In the coupled RTE–DA model, the domain \(\Omega\) is divided into RTE and DA sub-domains. The RTE is used as a forward model in sub-domain \(\Omega_{rte}\) in which the assumptions of the DA are not valid. The DA is used as a forward model in sub-domain \(\Omega_{da}\) which includes the remaining domain. Let \(\partial \Omega_{rte}\) be the boundary of the domain \(\Omega_{rte}\), and \(\partial \Omega_{da}\) be the boundary of the domain \(\Omega_{da}\). Furthermore, let \(\Gamma = \partial \Omega_{rte} \cap \partial \Omega_{da}\) denote the interface that separates the sub-domains \(\Omega_{rte}\) and \(\Omega_{da}\). The RTE and DA are coupled through boundary conditions on the interface \(\Gamma\). For the
external boundaries we use the following notations \( \partial \Omega_{\text{rte, out}} = \partial \Omega_{\text{rte}} \setminus \Gamma \) and \( \partial \Omega_{\text{da, out}} = \partial \Omega_{\text{da}} \setminus \Gamma \). In the frequency domain, the coupled RTE–DA model is of the form

\[
\frac{i \omega}{c} \phi(r, \hat{s}) + \hat{s} \cdot \nabla \phi(r, \hat{s}) + (\mu_s + \mu_a) \phi(r, \hat{s}) = \mu_s \int_{S_{n-1}} \Theta(\hat{s} \cdot \hat{s}') \phi(r, \hat{s}') d\hat{s}' + q(r, \hat{s}), \quad r \in \Omega_{\text{rte}} \quad (5)
\]

\[
\phi(r, \hat{s}) = \begin{cases} 
\phi_0(r, \hat{s}), & r \in \bigcup_j \varepsilon_j, \quad \hat{s} \cdot \hat{n} < 0 \\
0, & r \in \partial \Omega_{\text{rte, out}} \setminus \bigcup_j \varepsilon_j, \quad \hat{s} \cdot \hat{n} < 0 
\end{cases} \quad (6)
\]

\[
\phi(r, \hat{s}) = \frac{1}{|S_{n-1}|} \Phi(r) - \frac{n}{|S_{n-1}|} \hat{s} \cdot (\kappa \nabla \Phi(r)), \quad r \in \Gamma \quad (7)
\]

\[
-\nabla \cdot \kappa \nabla \Phi(r) + \mu_a \Phi(r) + \frac{i \omega}{c} \Phi(r) = q_0(r), \quad r \in \Omega_{\text{da}} \quad (8)
\]

\[
\Phi(r) + \frac{1}{2 \gamma_n} \kappa A \frac{\partial \Phi(r)}{\partial \hat{n}} = \begin{cases} 
\frac{I_s}{\gamma_n}, & r \in \bigcup_j \varepsilon_j \\
0, & r \in \partial \Omega_{\text{da, out}} \setminus \bigcup_j \varepsilon_j 
\end{cases} \quad (9)
\]

\[
\Phi(r) = \int_{S_{n-1}} \phi(r, \hat{s}) d\hat{s}, \quad r \in \Gamma. \quad (10)
\]

The FE-model for the coupled RTE–DA model was derived in [16] and it was further extended for light propagation in turbid medium with low-scattering and non-scattering regions in [14]. In this study, the FE-calculations are implemented accordingly.

3. RESULTS

The coupled RTE–DA model was tested with tissue-like optical properties in a realistic head geometry. The simulations were performed in a 2D segmented image that was done from a magnetic resonance image of a new-born infant head. The 2D head slice is shown in left image of Figure 1. The segmentation consisted of six domains which were: scalp, skull, grey matter, white matter, CSF around the brain, and CSF in ventricles. The optical parameters used in simulations are given in Table 1. The modulation frequency of the input signal was 100 MHz.

Figure 1: The left image: A segmented image from a new-born infant’s head. The right image: The coupled RTE–DA model computation domain; the CSF regions (dark grey colour), the RTE sub-domain (light grey colour), and the DA sub-domain (white colour). The location of the light source is marked with a triangle.

The forward problem was solved with the coupled RTE–DA model in the head slice domain. The computation domain is illustrated in the right image of Figure 1 where the CSF regions are marked with dark grey colour. The domain was divided into the RTE and DA sub-domains, \( \Omega_{\text{rte}} \) and \( \Omega_{\text{da}} \), which are illustrated in Figure 1 with light grey and white colours, respectively. In simulations, the source was located at the bottom of the computation domain. The location of the source is marked in Figure 1 with a small triangle. The coupled RTE–DA model was solved with the FEM with...
the streamline diffusion modification applied. For comparison, the FE-solutions of the RTE and the DA were computed in the whole domain. The FE-solutions were computed in the same spatial mesh as the FE-solution of the coupled RTE–DA model. In addition, the results were compared with Monte Carlo simulation. The Monte Carlo simulations were performed as in [9].

Table 1: The absorption coefficient $\mu_a$ and the scattering coefficient $\mu_s$ within the head slice.

<table>
<thead>
<tr>
<th></th>
<th>Scalp</th>
<th>Skull</th>
<th>Grey matter</th>
<th>White matter</th>
<th>CSF around the brain</th>
<th>CSF in the ventricles</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_a$ (mm$^{-1}$)</td>
<td>0.018</td>
<td>0.016</td>
<td>0.048</td>
<td>0.036</td>
<td>0.0048</td>
<td>0.0048</td>
</tr>
<tr>
<td>$\mu_s$ (mm$^{-1}$)</td>
<td>4.75</td>
<td>4</td>
<td>1.25</td>
<td>2.5</td>
<td>0.16</td>
<td>0.006</td>
</tr>
</tbody>
</table>

The photon densities inside the head slice are shown in Figure 2 which shows the FE-solutions of the RTE, the coupled RTE–DA model, and the DA (images from left to right in respective order). The logarithms of amplitudes are shown on the top row and the phase shifts are shown on the bottom row. The exitances on the boundary of the head slice are shown in Figure 3 which shows the exitances solved with the RTE, the coupled RTE–DA model, the DA, and Monte Carlo. The logarithms of amplitudes are shown on the left and the phase shifts are shown on the right. As it can seen from Figure 2, the photon densities solved with the RTE and the coupled RTE–DA model look similar. The DA solution differs from both of them. Examining the exitances on the boundary of the domain and comparing the FE-solutions with the Monte Carlo simulations show that the coupled RTE–DA model and the RTE give almost the same results as Monte Carlo. This can be seen from Figure 3. The DA solution differs from the other approaches clearly.

Figure 2: Logarithm of amplitude (top row) and phase shift (bottom row) of photon density within the head slice. From left to right: the RTE solution, the coupled RTE–DA model solution, and the DA solution.

Figure 3: Logarithm of amplitude (on the left) and phase shift (on the right) of exitance on the boundary of the head slice.

Information about FE-matrix sizes and number of non-zero elements in them as well as the computation times are given in Table 2. All the FE-solutions were computed using the biconjugate
gradient method with MATLAB® version 7.5 (R2007b), (The MathWorks, Inc.). The iterations were proceeded until they converged. As it can be seen from Table 2, the computation times for the RTE and the coupled RTE–DA model are both longer than for the whole domain DA. However, the coupled RTE–DA solution is obtained faster than the solution using the RTE in the whole domain. Moreover, the FE-matrix size and the number of non-zero elements are smaller in coupled RTE–DA model than in the whole domain RTE. This indicates that the coupled model requires less memory when solving the forward problem.

Table 2: FE-matrix sizes, number of non-zero elements, and the forward solution computation times for the RTE, the coupled RTE–DA model, and the DA.

<table>
<thead>
<tr>
<th></th>
<th>Matrix size</th>
<th>Non-zeros</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RTE:</td>
<td>351040 × 351040</td>
<td>77948928</td>
<td>14383</td>
</tr>
<tr>
<td>RTE–DA:</td>
<td>202002 × 202002</td>
<td>42099140</td>
<td>11509</td>
</tr>
<tr>
<td>DA:</td>
<td>10970 × 10970</td>
<td>76122</td>
<td>19.594</td>
</tr>
</tbody>
</table>

4. CONCLUSION

Utilizing the RTE in diffuse optical tomography was investigated. Light propagation was modelled with the coupled RTE–DA model, the RTE, the DA, and Monte Carlo in realistic two-dimensional head geometry from a new-born infant’s head. The results show that the coupled RTE–DA model gives almost the same results as the RTE and Monte Carlo. The DA solution, however, differs from the other approaches clearly. The results show that the RTE can be utilized in DOT in situations where the DA is not valid.

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REFERENCES


